Selected Solutions — Chapter 8

Investigate, page 470

1. b) The angles in each pair are supplementary.

d) The opposite angles of a cyclic quadrilateral are supplementary.

2. d) Each exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

8.1 Exercises, page 472

2. Explanations may vary. For part a:
The quadrilateral is cyclic, so the opposite angles add up to 180°. I set up the two equations that correspond to this fact and then solved them for x and y:

$$x + 110^\circ = 180^\circ$$
$$x = 70^\circ$$
$$y + 100^\circ = 180^\circ$$
$$y = 80^\circ$$

4. Explanations may vary: For part a:
I can determine x using the Angles in a Circle Theorem:

$$\begin{aligned} x &= \frac{1}{2}(360^\circ - 260^\circ) \\ &= \frac{1}{2}(100^\circ) \\ &= 50^\circ \end{aligned}$$

To determine y, I noticed that the angles with measures x and $(70^\circ + y)$ are opposite angles in a cyclic quadrilateral, so they add to 180°. I wrote down the equation that corresponds to this fact and then solved the resulting equation for y:

$$\begin{aligned} x + (70^\circ + y) &= 180^\circ \\ 50^\circ + (70^\circ + y) &= 180^\circ \\ 120^\circ + y &= 180^\circ \\ y &= 60^\circ \end{aligned}$$

5. a) The sum of x and y is 180° by a cyclic quadrilateral property. Thus, to find y when x is known, subtract x from 180°.

b) By a cyclic quadrilateral property, x and y are equal.

6. a) Construct OA, OC, and OD. OA = OC = OD, so the circle passes through all 3 points.

c) i) As B moves closer to the circle, ∠B becomes larger, until B reaches the circle, at which point ∠B and ∠D are supplementary. Angle D is constant.

ii) As B moves closer to the circle, ∠A and ∠C become smaller, until B reaches the circle, at which point ∠A and ∠C are supplementary.

e) Yes, the opposite angles in the cyclic quadrilateral are supplementary.

7. a) Both angles are subtended by the minor arc BD. From the Angles in a Circle Theorem,
∠DOB = 2∠DAB ①

Selected Solutions — Chapter 8

b) Both angles are subtended by the major arc BD. By the Angles in a Circle Theorem,
reflex $\angle DOB = 2\angle DCB$ ②

c) In parts a and b, we found expressions for $\angle DOB$ and reflex$\angle DOB$. These two angles add to 360°.
$\angle DOB + \text{reflex}\angle DOB = 360°$ ③
Substitute ① and ② into ③ and simplify.
$2\angle DAB + 2\angle DCB = 360°$
$\angle DAB + \angle DCB = 180°$
Therefore, Cyclic Quadrilateral Property 1 is proved.

8.2 Exercises, page 476

1. In $\triangle EAD$ and $\triangle ECB$
From the Corollary of the Cyclic Quadrilateral Theorem,
$\angle ADE = \angle CBE$
$\angle E$ is common.
From the Angles in a Triangle Theorem, since two pairs of corresponding angles are equal, the third pair of angles are equal.
$\angle DAE = \angle BCE$
Therefore, $\triangle EAD \sim \triangle ECB$

2. In $\triangle EAC$ and $\triangle EDB$
From Corollary 1 of the Angles in a Circle Theorem,
$\angle CAE = \angle BDE$
$\angle E$ is common.
From the Angles in a Triangle Theorem, since two pairs of corresponding angles are equal, the third pair of angles are equal.
$\angle ACE = \angle DBE$
Therefore, $\triangle EAD \sim \triangle ECB$

3. a) $\triangle PQR \cong \triangle RSR$ (SSS)
Since the triangles are congruent, corresponding angles are equal.
Thus, $\angle S = \angle Q$ ①
From the Cyclic Quadrilateral Theorem,
$\angle S + \angle Q = 180°$ ②
Substitute for $\angle S$ from equation ① into equation ②.
$2\angle Q = 180°$
$\angle Q = 90°$
But, $\angle S = \angle Q$, so $\angle S = 90°$
Thus, $\triangle PQR$ and $\triangle PSR$ are right triangles.

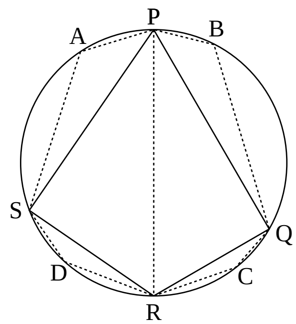
b) Quadrilateral PQRS is a square.

4. a) Consider cyclic quadrilateral QPAR.
From the Cyclic Quadrilateral Theorem,
$\angle A + \angle RQP = 180°$ ①
Similarly, in cyclic quadrilateral PBQR,
$\angle B + \angle PRQ = 180°$ ②
And, in cyclic quadrilateral PRCQ,
$\angle C + \angle RPQ = 180°$ ③
Add equations ①, ②, and ③.

Selected Solutions — Chapter 8

$\angle A + \angle B + \angle C + \angle RQP + \angle PRQ + \angle RPQ = 540^\circ$
But, from the Angles in a Triangle Theorem,
$\angle RQP + \angle PRQ + \angle RPQ = 180^\circ$
Therefore, $\angle A + \angle B + \angle C + 180^\circ = 540^\circ$
or, $\angle A + \angle B + \angle C = 360^\circ$

5. Problems may vary. Quadrilateral PQRS is inscribed in a circle. Points A, B, C, and D are any four points on the four arcs defined by the sides of the quadrilateral. Prove that
$\angle A + \angle B + \angle C + \angle D = 540^\circ$.



Consider the cyclic quadrilateral APRS.
From the Cyclic Quadrilateral Theorem,
$\angle A + \angle PRS = 180^\circ$ ①
Similarly, in cyclic quadrilateral PRQB,
$\angle B + \angle RPS = 180^\circ$ ②
Similarly, in cyclic quadrilateral PRCQ,
$\angle C + \angle RPQ = 180^\circ$ ③
And, in cyclic quadrilateral DSPR,
$\angle D + \angle SPR = 180^\circ$ ④
Add equations ①, ②, ③, and ④
$\angle A + \angle B + \angle C + \angle D + \angle PRS + \angle PRQ + \angle RPQ + \angle SPR = 720^\circ$
But from the Cyclic Quadrilateral Theorem,
$\angle PRS + \angle PRQ + \angle RPQ + \angle SPR = 180^\circ$
$\angle A + \angle B + \angle C + \angle D + 180^\circ = 720^\circ$
or, $\angle A + \angle B + \angle C + \angle D = 540^\circ$

7. $\angle DEC = 90^\circ$, so $\triangle DEC$ is a right triangle. Thus, $\cos \angle MDE = \frac{DE}{DC}$.
Use the cosine law in $\triangle DEM$.

$ME^2 = DE^2 + DM^2 - 2(DE)(DM) \cos \angle MDE$

$ME^2 = DE^2 + DM^2 - 2(DE)(DM)\frac{DE}{DC}$

$ME^2 = DE^2 + DM^2 - 2(DE)(DM)\frac{DE}{2DM}$

$ME^2 = DE^2 + DM^2 - DE^2$

$ME^2 = DM^2$

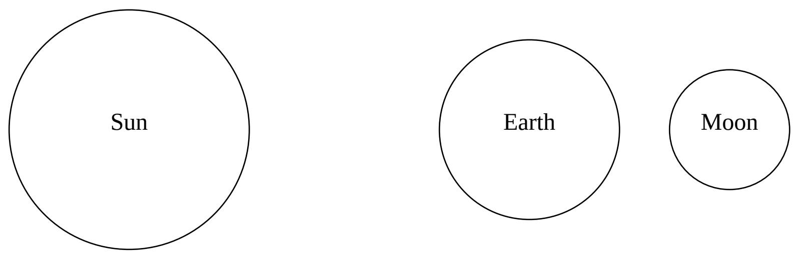
Thus, ME = DM
Thus, $\triangle DME$ is isosceles.
$\angle MDE = \angle MED$
From Corollary 1 of the Angles in a Circle Theorem,
$\angle MDE = \angle EAN$
Thus, $\angle MED = \angle EAN$ ①
From the Opposite Angles Theorem,

∠MEC = ∠AEN ②
∠MEC + ∠MED = ∠DEC = 90°
Substitute for ∠MEC and ∠MED from ① and ②.
∠AEN + ∠EAN = 90°
From the Angles in a Triangle Theorem in △EAN,
∠ENA = 180° – (∠EAN + ∠AEN)
= 180° – 90°
= 90°
Thus, MN is perpendicular to AB.

Linking Ideas: Mathematics & Science
Eclipses, page 478

3. The lines on the diagram represent the paths along which rays of light can travel from the Sun to Earth. The lines that end on the edge of the umbra outline a region on Earth that no rays from the Sun can reach. The lines that end on the edge of the penumbra outline a region on Earth where some rays from the Sun can reach but others are blocked by the moon.

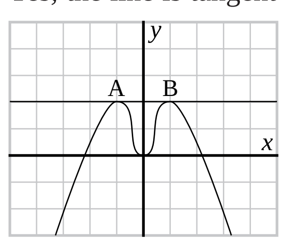
4.



6. Even though solar and lunar eclipses occur frequently, lunar eclipses are seen more frequently. That's because the darkened full moon can be seen from anywhere on the nighttime half of the Earth (where visibility is clear) during a lunar eclipse. To see a total solar eclipse, one has to be in the umbra, which covers a relatively small area.

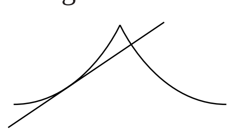
Mathematics File: What Is a Tangent?, page 480

2. **a)** Yes, the line is tangent to the curve at A and B.



b) The tangent line always intersects the curve at the point of tangency. However, the tangent line could also intersect the curve at other points at which it is not tangent; see the diagram at the right in the middle of page 480.

3.



Selected Solutions — Chapter 8

4. **a)** PQ becomes a better and better approximation to a tangent to the circle at P as Q approaches P. When Q and P coincide, PQ is a tangent to the circle.

b) We obtain the same result as in part a.

5. **b)** As Q moves counterclockwise, the angle gets smaller, is zero when POQ is a diameter, then increases to a maximum of 90˚.

c) As Q move clockwise, the angle increases to a maximum of 90˚.

Investigate, page 482

1. **c)** A tangent to a circle is perpendicular to the radius at the point of tangency.

2. **c)** The tangent segments to a circle from an external point are equal.

8.3 Exercises, page 485

3. Explanations may vary. For exercise 1, part a:
Since, AB is a tangent to the circle, ∠ABO = 90˚. Thus, △OBA is a right triangle, and the Pythagorean Theorem can be applied. I wrote the resulting equation and solved it for x:

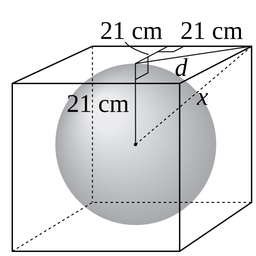
$$x^2 + 12^2 = 13^2$$
$$x^2 = 25$$
$$x = 5$$

15. **b)** △PAO and △PBO are right triangles, since ∠PAO and ∠PBO are right angles, from Tangent Property 1.

c) Apply the Pythagorean Theorem to each right triangle.
In △PAO: $PA^2 = PO^2 - OA^2$
In △PBO: $PB^2 = PO^2 - OB^2$
But OA = OB because they are radii
Hence, $PA^2 = PB^2$, and PA = PB

16. Construct a right triangle on the top surface of the box, joining the top of the globe to a corner of the box and to the midpoint of a side. Then use the Pythagorean Theorem to determine the distance d between the top of the globe and a corner of the box:

$$d^2 = 21^2 + 21^2$$
$$d^2 = 882$$



Now construct another right triangle, this one joining the centre of the sphere with the top of the globe and the corner of the box. Then the distance x from the centre of the sphere to the corner of the box can also be calculated using the Pythagorean Theorem:

$x^2 = 21^2 + d^2$
$x^2 = 441 + 882$
$x^2 = 1323$
$x \doteq 36.4$
The centre of the globe is approximately 36.4 cm from a vertex of the box.

8.4 Exercises, page 492

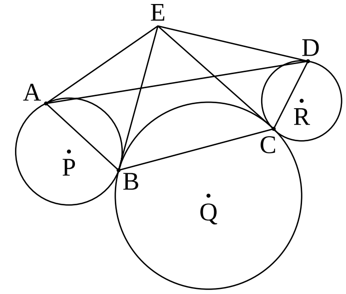
1. From the Equal Tangents Theorem, PQ = PR
Thus, $\triangle$PQR is isosceles.
Hence, $\angle$PQR = $\angle$PRQ

2. **a)** AB = AP + BP
From the Equal Tangents Theorem,
AP = AS
BP = BQ
Hence, AB = AS + BQ

b) CD = CR + DR
From the Equal Tangents Theorem,
CR = CQ
DR = DS
Hence, CD = CQ + DS

3. From the Tangent-Radius Theorem,
$\angle$PBO = 90˚ and $\angle$PAO = 90˚
Therefore, $\angle$PBO + $\angle$PAO = 180˚
Since opposite angles are supplementary,
PAOB is a cyclic quadrilateral.

4. To prove that ABCD is a cyclic quadrilateral, show that opposite angles in the quadrilateral add to 180˚.
From the Equal Tangents Theorem,
EA = EA = EC = ED
From the Isosceles Triangle Theorem,

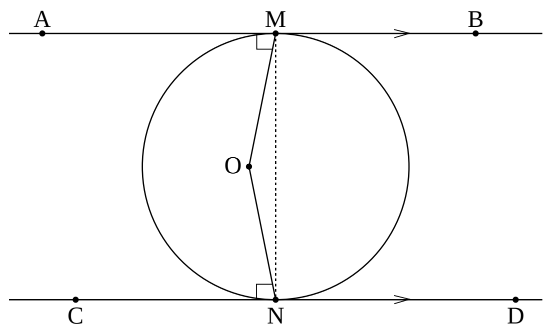


$\angle$EAD = $\angle$EDA
$\angle$EAB = $\angle$EBA
$\angle$EBC = $\angle$ECB
$\angle$EDC = $\angle$ECD
$\angle$BAD = $\angle$EAB – $\angle$EAD
= $\angle$EAB – $\angle$EDA
$\angle$BCD = $\angle$ECD + $\angle$ECB
= $\angle$EDC + $\angle$ECB
$\angle$BAD + $\angle$BCD = $\angle$EAB – $\angle$EDA + $\angle$EDC + $\angle$ECB
= $\angle$EAB + $\angle$ECB + $\angle$EDC – $\angle$EDA
= $\angle$EAB + $\angle$ECB + $\angle$ADC
= $\angle$EBA + $\angle$EBC + $\angle$ADC
= $\angle$ABC + $\angle$ADC
But $\angle$BAD + $\angle$BCD + $\angle$ABC + $\angle$ADC = 360˚, since they are the angles in a quadrilateral

Selected Solutions — Chapter 8

Thus, $2(\angle BAD + \angle BCD) = 360^\circ$
$\angle BAD + \angle BCD = 180^\circ$
Thus, opposite angles are supplementary, so ABCD is a cyclic quadrilateral.

5. Let O be the centre of the circle. Since AB and CD are tangents, $AB \perp OM$, and $CD \perp ON$. To prove that MN is a diameter, we must show that O lies on MN.
Use indirect proof. Assume that O does not lie on MN, as in the diagram.



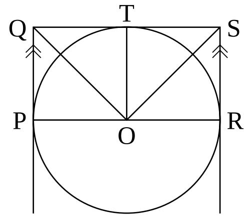
Then $\angle BMN = \angle CNM$ since they are alternate angles. But from the diagram, $\angle BMN < 90^\circ$, and $\angle CNM < 90^\circ$. This contradicts the deduction that $\angle BMN = \angle CNM$. Thus, the assumption that O does not lie on MN must be false. Therefore, O must lie on MN, and so MN must be a diameter.

6. The perimeter of $\triangle PQR$ is $PR + RQ + QP$.
We can write RQ as $PX + QX$.
From the Equal Tangents Theorem, we know that $RX = RS$ and $QX = QT$.
Substitute these lengths in the perimeter expression.
Perimeter $= PR + RS + QT + QP$.
But $PR + RS = PS$ and $QT + QP = PT$
Thus, perimeter $= PS + PT$
From the Equal Tangents Theorem, $PS = PT$
Therefore, the perimeter is 2PS.

7. Since OQ and OR are radii, $OQ = OR$.
We are given $OQ = QR$
Hence $QR = OR$, and $\triangle ORQ$ is equilateral.
Therefore, $\angle OQR = 60^\circ$
Since $\angle OQP = 90^\circ$, then $\angle RQP = 90^\circ - 60^\circ$, or 30°
Since $\angle PRO$ is a straight angle, $\angle PRQ = 180^\circ - \angle ORQ$
$= 180^\circ - 60^\circ$
$= 120^\circ$
Use the Angles in a Triangle Theorem in $\triangle PRQ$.
Then $\angle RPQ = 30^\circ$
Since $\angle RPQ = \angle RQP$, then $\triangle PRQ$ is isosceles with $RP = RQ$.
But we know $RQ = OR$, hence $RP = OR$.

Selected Solutions — Chapter 8

8. Label T, the point of tangency of QS.
Join OP, OR, and OT.



From the Equal Tangents Theorem, QP = QT and ST = SR
Radii are equal: OP = OT = OR

In △QPO and △QTO
QP = QT
QO is common.
OP = OT
Therefore, △QPO ≅ △QTO (SSS)
Since the triangles are congruent, corresponding angles are equal:
∠PQO = ∠TQO = x

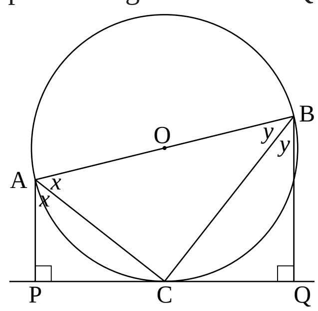
In △TSO and △RSO
TS = RS
OT = OR
OS is common.
Therefore, △TSO ≅ △RSO (SSS)
Since the triangles are congruent, corresponding angles are equal:
∠TSO = ∠RSO = y
Since QP ∥ SR, interior angles are supplementary:
∠PQT + ∠TSR = 180°
But ∠PQT = $2x$ and ∠TSR = $2y$
Therefore, $2x + 2y = 180°$
$x + y = 90°$
In △QOS, using the Angles in a Triangle Theorem,
∠QOS = 180° − $(x + y)$
= 180° − 90°
= 90°
Hence, △OQS is a right triangle.

9. **a)** Construct radius OC.
From the Tangent-Radius Theorem,
∠OCD = 90°
OA = OC, since they are radii
Hence, △OAC is isosceles.
Thus, ∠OAC = ∠OCA
∠OCA + ∠ACD = 90° ①
From the Angles in a Triangle Theorem in △ADC,
∠DAC + ∠ACD = 90° ②
Comparing ① and ②,
∠OCA = ∠DAC
Thus, ∠DAC = ∠OAC
Therefore, AC bisects ∠BAD.

Selected Solutions — Chapter 8

b) OC = OB, since they are radii
Hence, △OCB is isosceles.
Thus, ∠OCB = ∠OBC
But, ∠OBC = ∠CBE
Therefore, ∠OCB = ∠CBE
From the converse of the Parallel Lines Theorem,
OC || BE
Since these segments are parallel, corresponding angles are equal,
∠OCE = ∠BEN
Since ∠OCE = 90°, then ∠BEN = 90°
Thus, BE is perpendicular to MN.

10. What we must show is that no matter where the line PQ is a tangent to the circle, the perpendicular distances from A to PQ and from B to PQ have the same sum. For simplicity, no matter where the line PQ is, let P be the point on PQ such that AP ⊥ PQ, and let Q be the point on PQ such that BQ ⊥ PQ. Then it is required to show that AP + BQ is constant; that is, AP + BQ is independent of the positioning of the line PQ.



Let O be the centre of the circle and let C be the point of tangency. Since AP is perpendicular to PQ, and BQ is perpendicular to PQ, it follows from exercise 9a, that AC bisects ∠PAO and BC bisects ∠QBO.
From the proof in exercise 9a,
∠PAC = ∠CAB = x
∠ABC = ∠CBQ = y
From the Semicircle Theorem, ∠ACB = 90°
From the Angles in a Triangle Theorem,
In △ACB: 90° + x + y = 180°
Therefore, $x + y = 90°$ ①
In △APC: 90° + x + ∠ACP = 180°
x + ∠ACP = 90° ②
Comparing ① and ②, ∠ACP = y
In △CBQ: 90° + y + ∠BCQ = 180°
y + ∠BCQ = 90° ③
Comparing ① and ③, ∠BCQ = x
Since these three triangles have corresponding angles equal, the triangles are similar:
△APC ~ △ACB ~ △CQB
Corresponding sides are in the same ratio:

Selected Solutions — Chapter 8

$$\frac{PA}{CA} = \frac{CA}{BA} \text{ or } PA = \frac{CA^2}{BA}$$

$$\frac{QB}{CB} = \frac{CB}{AB} \text{ or } QB = \frac{CB^2}{AB}$$

$$PA + QB = \frac{CA^2}{BA} + \frac{CB^2}{AB}$$

$$PA + QB = \frac{CA^2 + CB^2}{AB}$$

From the Pythagorean Theorem in $\triangle ABC$, $CA^2 + CB^2 = AB^2$

Therefore, $PA + QB = \frac{AB^2}{AB}$

$= AB$

Mathematical Modelling: How Long Is the Chain?, page 494

1. a) The distance between the centres of the circles is 87 cm. The length of chain around the small circle is half of the circumference. This is $\pi \times 9 \doteq 28.274$. The length of chain around the second circle is $\pi \times 19 \doteq 59.690$. The total length of bicycle chain is $2 \times 87 + 28.274 + 59.690 \doteq 261.964$. This is approximately 261.96 cm.

b) The difference is (263.12 – 261.96) cm = 1.16 cm.

c) As a percent, it is $\frac{1.16}{263.12} \times 100\% \doteq 0.44\%$

2. a) No. Since the diameters of the circles are larger, the approximation that the length of chain around the circles is approximately half the circumference of each circle is not as accurate as before. Also, since the circles are closer together, and larger, the approximation that the length of chain along a common tangent segment is equal to the distance between the centres of the circles is not as accurate as before.

b) For the alternator belt, the length in centimetres is $2 \times 54.0 + \pi \times 28.5 + \pi \times 16.5 = 249.37$. The difference in lengths is (252.05 – 249.37) cm = 2.68 cm. The percent difference is

$\frac{2.68}{252.05} \times 100\% \doteq 1.06\%$

3. The lengths of the tangents are longer than the distance between the centres of the circles. The arcs AD and BC are not equal to half the circumference.

4. Diagrams and answers may vary.

5. Answers may vary. No diagram is a perfect representation of the original. Also every measurement has uncertainty.

6. Answers may vary. The thickness of the belt or chain must be taken into account. Bicycle gear wheels have teeth on them, so it is not clear what distance should be used for the radius of the wheel; it depends on how the chain conforms to the wheel.

7. a) Find the midpoint of PQ. Draw a circle with the midpoint as the centre and $\frac{1}{2}$PQ as the radius. The point of intersection of this new circle with the given circle is E. (PQ is the diameter of the new circle, and ∠PEQ is an angle inscribed in the new semicircle, so it is a right angle.

b) AEQB is a rectangle, so ∠EAB = ∠QBA = 90°. Since QB is a radius, AB must be a tangent.

8. a) Triangle PEQ is a right triangle since ∠PEQ = 90°.

$$\begin{aligned}PE &= PA - EA\\ &= 28.5\text{ cm} - 16.5\text{ cm}\\ &= 12.0\text{ cm}\end{aligned}$$

Using the Pythagorean Theorem in △PQE,

$$PQ^2 = EQ^2 + PE^2$$

$$\begin{aligned}EQ &= \sqrt{PQ^2 - PE^2}\\ &= \sqrt{54.0^2 - 12.0^2}\\ &\doteq 52.65\end{aligned}$$

b) ABQE is a rectangle since EA = BQ and the angles are all 90°. Therefore, EQ = AB

9. a)

$$\begin{aligned}\cos P &= \frac{PE}{PQ}\\ &= \frac{12.0}{54.0}\\ &= 0.\overline{2}\end{aligned}$$

$$\angle P \doteq 77.16^\circ$$

b) Extend PQ to a point R.
Since AP || BQ, and PQR is a transversal, corresponding angles are equal:
∠APQ = ∠BQR ①
Similarly, DP || CQ and ∠DPQ = ∠CQR ②
Add ① and ②. ∠APQ + ∠DPQ = ∠BQR + ∠CQR
Therefore, ∠APD = ∠BQC
Now for the measures of the angles: ∠APD = 2∠EPQ
≐ 154.32°

Since ∠BQC = ∠APD, ∠BQC ≐ 154.32°

c) The fraction of the second circle represented by the minor arc is

$$\frac{\angle BQC}{360^\circ} \doteq \frac{154.32^\circ}{360^\circ} \doteq 0.4287$$

The length of the belt from B to C is the fraction above multiplied by the circumference of the circle.

$$0.4287 \times 2\pi \times 16.5 \doteq 44.441$$

The belt is about 44.441 cm long.

d) The fraction of the first circle represented by the major arc is 1 – 0.4287 = 0.5713.
The length of the belt from A to D is

$$0.5713 \times 2\pi \times 28.5 \doteq 102.303$$

The belt is about 102.303 cm long.

10. The length of the alternator belt is

$$102.303 + 44.441 + 2 \times 52.65 \doteq 252.04$$

The alternator belt is about 252.04 cm long.

Selected Solutions — Chapter 8

11. This is so AEQB will be a rectangle; this is what makes the calculation work.

Investigate, page 497

1. d) The angle between a tangent to a circle and a chord of the circle is equal to the inscribed angle on the opposite side of the chord.

8.5 Exercises, page 500

3. Explanations may vary. For part a:
First I used the Tangent-Chord Theorem to relate the known angle to an angle in $\triangle$QMN:
$\angle QNM = \angle PQM = 75^\circ$
Then I noticed that since $\triangle$QMN is isosceles,
$\angle MQN = \angle MNQ = 75^\circ$.
Finally, the angles in the triangle add to 180°; I wrote the equation that corresponds to this fact and solved it for x.

$$\begin{aligned}\angle NMQ + \angle QNM + \angle NQM &= 180^\circ\\ x + 75^\circ + 75^\circ &= 180^\circ\\ x &= 180^\circ - 2(75^\circ)\\ &= 30^\circ\end{aligned}$$

Thus, $x = 30^\circ$

5. Explanations may vary. For part a:
First I noticed that by the Tangent-Chord Theorem, $x = y$, since x is the inscribed angle opposite y. Then I noticed that x and 110° are opposite angles in a cyclic quadrilateral, so they must add up to 180°.
I wrote the corresponding equation and solved for x:

$$\begin{aligned}x + 110^\circ &= 180^\circ\\ x &= 180^\circ - 110^\circ\\ x &= 70^\circ\end{aligned}$$

Since $x = y$,
$y = 70^\circ$

6. a) This is true by the Tangent-Chord Theorem.

b) One way is to use the Tangent-Chord Theorem.
Here is another way.
From part a, $\angle ABC = \angle ACE$
Since $\angle DCE$ is a straight angle,

$$\begin{aligned}\angle BCD &= 180^\circ - \angle ACE - \angle ACB\\ &= 180^\circ - \angle ABC - \angle ACB\end{aligned}$$

From the Angles in a Triangle Theorem in $\triangle$ABC,
$\angle BCD = \angle BAC$

7. No, it is not possible for ABCD to be a cyclic quadrilateral.
Join AC.
From the Tangent-Chord Theorem,
$\angle DAC = \angle ABC$
$\angle ACD = \angle ABC$
Therefore, $\angle DAC = \angle ACD$

Selected Solutions — Chapter 8

From the Angles in a Triangle Theorem in $\triangle ADC$,
$\angle ADC = 180^\circ - 2\angle DAC$
$= 180^\circ - 2\angle ABC$
$\angle ADC + 2\angle ABC = 180^\circ$
Therefore, $\angle ADC + \angle ABC \neq 180^\circ$, and ABCD is not a cyclic quadrilateral.

13. Since MN || AC, alternate angles are equal:
$\angle CBN = \angle MNB$
From the Tangent-Chord Theorem,
$\angle CBN = \angle BMN$
Thus, $\angle MNB = \angle BMN$
Hence, $\triangle MBN$ is isosceles.

14. From the Equal Tangents Theorem, PA = PB
Thus, $\triangle PAB$ is isosceles and $\angle PBA = \angle PAB = x$
From the Tangent-Chord Theorem,
$\angle ACB = \angle PAB = x$
But since AB and AC are equidistant from O, AB = AC and $\triangle ABC$ is isosceles.
Therefore $\angle ABC = \angle ACB = x$
From the Angles in a Triangle Theorem
In $\triangle PAB$: $\angle APB + 2x = 180^\circ$ ①
In $\triangle ABC$: $\angle BAC + 2x = 180^\circ$ ②
Comparing ① and ②, $\angle APB = \angle BAC$

15. From the Tangent-Radius Theorem,
$\angle PAO = 90^\circ$ and $\angle PBO = 90^\circ$
Therefore, quadrilateral PAOB is cyclic, since opposite angles are supplementary.
Consider the other pair of opposite angles.
$\angle APB + \angle AOB = 180^\circ$
Multiply by 2.
$2\angle APB + 2\angle AOB = 360^\circ$
But $\angle AOB + \text{reflex } \angle AOB = 360^\circ$
Therefore, $2\angle APB + 2\angle AOB = \angle AOB + \text{reflex } \angle AOB$
$2\angle APB = \text{reflex } \angle AOB - \angle AOB$
$\angle APB = \frac{1}{2}(\text{reflex } \angle AOB - \angle AOB)$

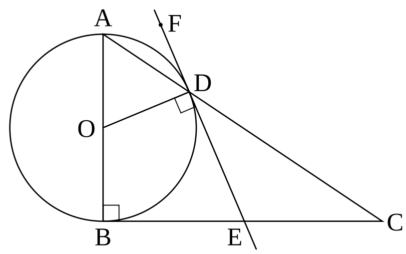
16. a) Let $\angle QAB = x$
From the Tangent-Radius Theorem,
$\angle OAB = 90^\circ$
$\angle OAQ = \angle OAB - \angle QAB$
$= 90^\circ - x$
Since radii are equal, OA = OQ
Then $\triangle OQA$ is isosceles, with $\angle OQA = \angle OAQ$
Hence, $\angle OQA = 90^\circ - x$
From the Angles in a Triangle Theorem,
$\angle QOA = 180^\circ - \angle OAQ - \angle OQA$
$= 180^\circ - 2(90^\circ - x)$
$= 2x$
$= 2\angle QAB$

Selected Solutions — Chapter 8

From the Angles in a Circle Theorem,
∠QOA = 2∠QRA
Thus, 2∠QAB = 2∠QRA
∠QAB = ∠QRA
This proves the Tangent-Chord Theorem.

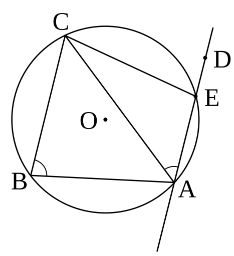
b) Answers may vary.

17. In circle, centre O, AB is a diameter and BC is a tangent. The circle intersects AC at D. Tangent FD is drawn and intersects BC at E.
We want to prove that BE = EC.
Join OD. Then, from the Tangent-Radius Theorem,
∠ODF = ∠ODE = 90°



Since radii are equal, OA = OD
Therefore, △AOD is isosceles with ∠OAD = ∠ODA = x
From the Angles in a Triangle Theorem in △OAD,
∠AOD = 180° – 2x ①
Since ∠ODE = ∠OBE = 90°, then from the converse of the Cyclic Quadrilateral Theorem, OBED is a cyclic quadrilateral.
From the Corollary of the Cyclic Quadrilateral Theorem,
∠AOD = ∠BED ②
Comparing ① and ②, ∠BED = 180° – 2x
Since ∠BEC is a straight angle, ∠DEC = 2x
Since ∠ODF = 90°, then ∠FDA = 90° – x
Using the Opposite Angles Theorem, ∠CDE = ∠FDA = 90° – x
Using the Angles in a Triangle Theorem in △DCE,
∠DCE = 180° – 2x – (90 – x)
∠DCE = 90° – x
Since ∠CDE = ∠DCE, △DEC is isosceles with ED = EC ③
From the Equal Tangents Theorem, ED = EB ④
Comparing ③ and ④, EC = EB
The tangent at D bisects the leg BC.

18. The converse of the Tangent-Chord Theorem: If △ABC is inscribed in a circle, centre O, so that ∠ABC is equal to ∠DAC, where AD is a line, then AD is a tangent to the circle.



Selected Solutions — Chapter 8

Use indirect proof.
Assume AD is not tangent to the circle.
Then, AD cuts the circle at another point, E.
Connect E to C.
Since ABCE is a cyclic quadrilateral, $\angle ABC + \angle CEA = 180°$.
Rearranging, $\angle CEA = 180° - \angle ABC$ ①
Also, $\angle EAC = \angle ABC$ ②
In $\triangle CEA$, $\angle CEA + \angle EAC + \angle ACE = 180°$ ③
Substituting ① and ② into ③,

$$180° - \angle ABC + \angle ABC + \angle ACE = 180°$$
$$\angle ACE = 0°$$

This contradicts the given information that there exists a point E to form $\triangle EAC$.
Hence, the assumption that AD is not tangent is incorrect.
Therefore, AD is tangent to the circle.

Problem Solving: Sweeping a Circle with Lines, page 504

1. **a)** Both triangles have $\angle APQ$ in common. Since the line PQ is a tangent to the circle, we can use the Tangent-Chord Theorem. It follows that $\angle PBQ = \angle PQA$. From the Angles in a Triangle Theorem, since two pairs of corresponding angles are equal, the third pair of angles must be equal. Therefore, $\triangle PAQ \sim \triangle PBQ$.

b) Since the triangles are similar, corresponding sides are in the same ratio.

$$\frac{PQ}{PB} = \frac{PA}{PQ}$$

or

$$PQ^2 = PA \times PB$$

c) Using the result from part b,

$$PQ^2 = PA_1 \times PB_1$$

and

$$PQ^2 = PA_2 \times PB_2$$

Therefore

$$PA_1 \times PB_1 = PA_2 \times PB_2$$

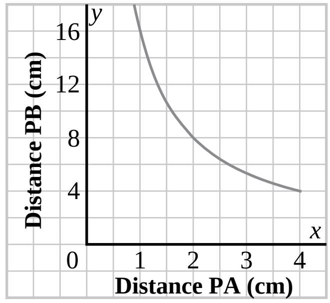
The product of their lengths is equal.

2. At Q, $PA = PB$ (and we call them PQ). Between Q and R, $PA \times PB = PQ^2$. At R, $PA = PB$ (and we call them both PR).

3. Answers may vary.

4. **a)** The graph looks like part of a hyperbola.

b) For the diagram shown, PQ is about 4 cm, and the radius of the circle is 7.5 cm.



5. **a)** The domain would be larger or smaller.

b) The domain is shifted to the right or to the left, and the graph is stretched or compressed.

c) There is no graph.

7. If P is inside the circle, PQ is not a tangent, and we cannot use the Tangent-Chord Theorem. However, if two chords are AB and CD, then PA × PB = PC × PD.

8.6 Exercises, page 511

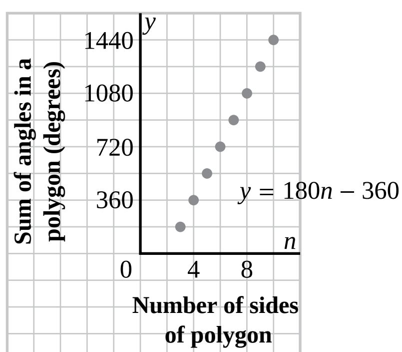
5. Explanations may vary. For part a of exercises 4 and 5:
For the 5-sided polygon, I used a formula developed in this section to determine the sum of the angles:

$$\text{sum of angles} = 180^\circ(5) - 360^\circ$$
$$= 540^\circ$$

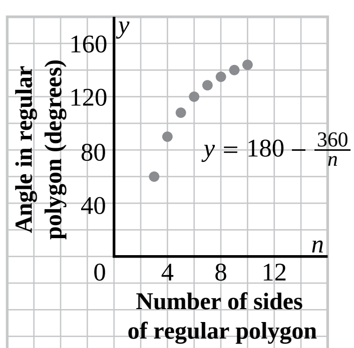
Then I used another formula to determine the measure of each angle:

$$\text{measure of each angle} = 180^\circ - \frac{360^\circ}{5}$$
$$= 108^\circ$$

6. **b)** **ii)**



7. **c)**



Selected Solutions — Chapter 8

8. b) Explanations may vary. For part i:
I counted the number of sides, n, then used the formula sum of angles = $180n - 360$, with $n = 6$.
Sum of angles is $180^\circ(6) - 360^\circ = 720^\circ$
Another way is to divide the polygon into triangles, by drawing diagonals from one vertex. There are 4 triangles, so the angle sum is $4 \times 180^\circ = 720^\circ$

13. The exterior angle of a regular polygon is $\frac{360^\circ}{n}$. The internal angle of a regular polygon is $180^\circ - \frac{360^\circ}{n}$. For regular polygons whose interior angles are natural number multiples of their exterior angles,
$180^\circ - \frac{360^\circ}{n} = p \times \frac{360^\circ}{n}$, where p is a natural number.
Multiply both sides of the previous equation by $\frac{n}{360^\circ}$ to obtain
$\frac{n}{2} - 1 = p$
$n = 2(p + 1)$
Since p is a natural number, all polygons with an even number of sides have this property. Here is a list of the first few possible pairs of p and n:

p	n
1	4
2	6
3	8
4	10
5	12
6	14

14. a) The middle third of each straight line is cut out, and two lines of length equal to one-third the original side length are inserted to make a new vertex.

b) The first two are easy to count, but the last two are tricky. We need to find a pattern. The first polygon has 3 sides and the second polygon has 12 sides. For each original side, four new ones appear. So the third polygon has 48 sides and the fourth 192.

c) The interior angles are all either 60° or 240°.

d) The first polygon is a triangle and has a measure of 180° for its interior angles. The second polygon has the same angles as the first, plus three new ones for each side of the previous polygon.
$180^\circ + 3(240^\circ + 60^\circ + 240^\circ) = 1800^\circ$
The pattern is similar for the others. The third polygon has the same angles as the second plus three new ones for each side of the second polygon.
$1800^\circ + 12(2 \times 240^\circ + 60^\circ) = 8280^\circ$
Similarly for the fourth
$8280^\circ + 48(2 \times 240^\circ + 60^\circ) = 34\ 200^\circ$

e) No. Because the process of adding sides goes on indefinitely, the resulting geometric figure does not have a finite number of sides.

Selected Solutions — Chapter 8

Linking Ideas: Mathematics & History
A Bestseller from Way Back, page 514

Lists may vary.

Opposite Angles Theorem, Isosceles Triangle Theorem, Parallel Lines Theorem, Angles in a Triangle Theorem, Exterior Angle Theorem, Congruent Triangle Theorems (SSS, SAS, ASA), Perpendicular Bisector Theorem, Angle Bisector Theorem, Pythagorean Theorem, Chord Perpendicular Bisector Theorem, Two Chords Theorem, Angles in a Circle Theorem, Semicircle Theorem, Cyclic Quadrilateral Theorem, Tangent-Radius Theorem, Tangent-Diameter Theorem, Equal Tangents Theorem, Tangent-Chord Theorem, Angle Sum Theorem for Polygons

8 Review, page 515

2. **a)** ABCD is a parallelogram because opposite sides are parallel.

b) If opposite sides remain parallel, the only cyclic figure is a rectangle, when opposite angles are supplementary.

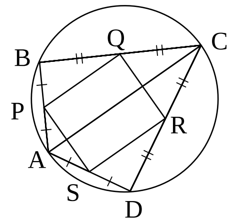
c) A cyclic parallelogram is a rectangle.

3. From Corollary 2 of the Chord Perpendicular Bisector Theorem,
$\angle ODA = 90^\circ$
$\angle OEA = 90^\circ$
Thus, opposite angles are supplementary, and ADOE is cyclic.

4. In $\triangle ADE$ and $\triangle ABC$,
$AD = \frac{1}{2}AB$
$AE = \frac{1}{2}AC$
$\angle A$ is common.
Since 2 pairs of corresponding sides are in the same ratio, and their contained angles are equal,
$\triangle ADE \sim \triangle ABC$
Since the triangles are similar, their corresponding angles are equal.
Thus, $\angle ADE = \angle ABC$ and $\angle AED = \angle ACB$
From the converse of the Parallel Lines Theorem, since corresponding angles are equal, DE is parallel to BC.
Since the lines are parallel, interior angles are supplementary,
$\angle DBC + \angle BDE = 180^\circ$ and $\angle DEC + \angle ECB = 180^\circ$
$\triangle ABC$ is isosceles.
Thus, $\angle DBC = \angle ECB$
Thus, $\angle ECB + \angle BDE = 180^\circ$ and $\angle DEC + \angle DBC = 180^\circ$
Since opposite angles are supplementary, BCED is cyclic.

5. In the diagram, ABCD is a cyclic quadrilateral. The midpoints of the sides are P, Q, R, and S. We want to prove that PQRS is a rectangle.
Join AC.

Selected Solutions — Chapter 8



In the $\triangle ABC$ and $\triangle ADC$

$AB = AD$
$BC = DC$
AC is common.

Therefore, $\triangle ABC \cong \triangle ADC$ (SSS)
Since the triangles are congruent, corresponding angles are equal,
$\angle ABC = \angle ADC$ ①
Since ABCD is a cyclic quadrilateral, opposite angles are supplementary,
$\angle ABC + \angle ADC = 180^\circ$ ②
Comparing ① and ②, $\angle ABC = \angle ADC = 90^\circ$.
Since $\triangle ABC \cong \triangle ADC$,
$\angle BAC = \angle DAC = a$
$\angle BCA = \angle DCA = b$
In $\triangle PBQ$ and $\triangle ABC$

$PB = \frac{1}{2}AB$

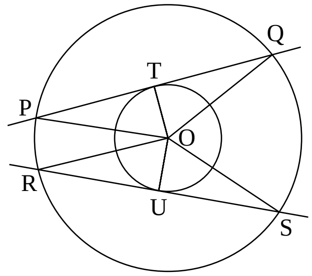
$PQ = \frac{1}{2}AC$

$\angle B$ is common.

Since two pairs of corresponding sides are in the same ratio and their contained angles are equal, $\triangle PBQ \sim \triangle ABC$.
Since the triangles are similar, their corresponding angles are equal.
$\angle BPQ = \angle BAC = a$
$\angle BQP = \angle BCA = b$
Since $PA = AS$, $\triangle PAS$ is isosceles with $\angle APS = \angle ASP$
Using the Sum of the Angles Theorem,
$\angle APS + \angle ASP + 2a = 180^\circ$
$\angle APS + a = 90^\circ$
But, from $\triangle PBQ$, $a + b = 90^\circ$
Therefore, $\angle APS = b$ and $\angle ASP = b$
Since $CQ = CR$, $\triangle CQR$ is isosceles, with $\angle CQR = \angle CRQ$.
Using the Sum of the Angles Theorem,
$\angle CQR + \angle CRQ + 2b = 180^\circ$
$2\angle CQR + 2b = 180^\circ$
$\angle CQR + b = 90^\circ$
But, $a + b = 90^\circ$
Therefore, $\angle CQR = a$ and $\angle CRQ = a$
At each vertex, P, Q, R, and S, there are 3 angles that form the straight angle. In each case, two angles are a and b. Hence, the third angle is 90°. That is, $\angle P = \angle Q = \angle R = \angle S = 90^\circ$.
Therefore, PQRS is a rectangle.

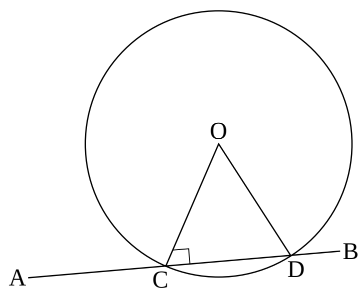
Selected Solutions — Chapter 8

6. Construct OP, OT, OQ, OR, OU, and OS.



From the Tangent-Radius Theorem, ∠OUR = 90˚
Use the Pythagorean Theorem in △OUR.
$RU^2 = OR^2 - OU^2$ ①
From the Tangent-Radius Theorem, ∠PTO = 90˚
Use the Pythagorean Theorem in △PTO.
$PT^2 = OP^2 - OT^2$ ②
Since radii are equal, OR = OP and OU = OT ③
Comparing ①, ②, and ③, $RU^2 = PT^2$ and RU = PT ④
It can similarly be shown that SU = QT ⑤
Adding ④ and ⑤ RU + SU = PT + QT
or RS = PQ

7. The converse of the Tangent-Radius Theorem: A straight line drawn at right angles to a radius of a circle at a point on the circle is a tangent to the circle.
Use indirect proof. Assume AB is not a tangent to the circle.
Then AB must intersect the circle at another point, D. Connect OD.



Radii are equal: OC = OD
Thus, △OCD is isosceles.
∠OCD = ∠ODC
But ∠ODC = 90˚
This is impossible, since there cannot be 2 right angles in a triangle.
Thus, the assumption is false, and AB is a tangent.

8. From the Equal Tangents Theorem, PB = PA
In △APQ and △BPQ,
PA = PB
PQ is common.
∠APQ = ∠BPQ
Thus, △APQ ≅ △BPQ (SAS)
Since the triangles are congruent, corresponding sides are equal,
AQ = BQ
And corresponding angles are equal, ∠AQP = ∠BQP

Selected Solutions — Chapter 8

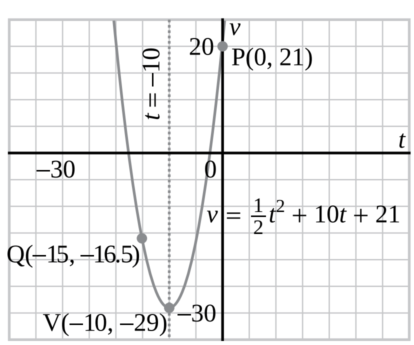
Therefore, 180˚ – ∠AQP = 180˚ – ∠BQP
And, ∠AQC = ∠BQC
In △AQC and △BQC,
AQ = BQ
∠AQC = ∠BQC
QC is common.
Thus, △AQC ≅ △BQC (SAS)
Since the triangles are congruent, corresponding angles are equal.
Hence, ∠CAQ = ∠CBQ
From the Cyclic Quadrilateral Theorem, ∠CAQ + ∠CBQ = 180˚
Hence, ∠CAQ = 90˚

9. We want to prove that RE = RF, PE = PD, and QD = QF.
We cannot use the Equal Tangents Theorem.
From the Tangent-Chord Theorem:
∠RFE = ∠FDE, and
∠REF = ∠FDE
Therefore, ∠RFE = ∠REF
Hence, △REF is isosceles, with RE = RF.
From the Tangent-Chord Theorem:
∠PED = ∠EFD
∠PDE = ∠EFD
Therefore, ∠PED = ∠PDE
Hence, △PED is isosceles with PE = PD.
From the Tangent-Chord Theorem:
∠QDF = ∠DEF
∠QFD = ∠DEF
Therefore, ∠QDF = ∠QFD
Hence, △QFD is isosceles, with QD = QF.

8 Cumulative Review, page 517

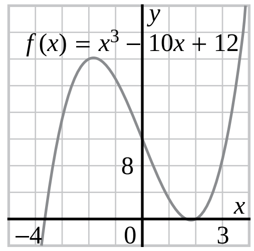
9. Answers may vary: For part a:
The general equation of a parabola is $y = a(x - h)^2 + k$, where (h, k) is the vertex. In this case, $h = -1$ and $k = 4$, so I substituted these values into the equation to get $y = a(x + 1)^2 + 4$. But the y-intercept is 16, so I substituted this value to get $16 = a + 4$ and $a = 12$. So the equation is $y = 12(x + 1)^2 + 4$.

10.



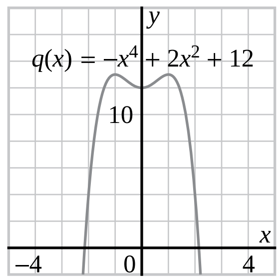
Selected Solutions — Chapter 8

11. a)



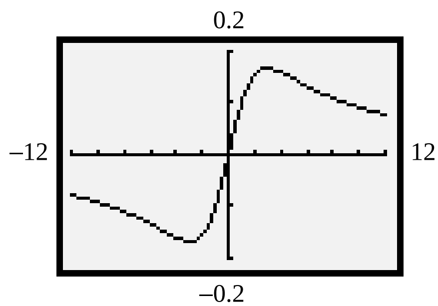
The zeros are approximately −3.6, 1.6, and 2.

b)

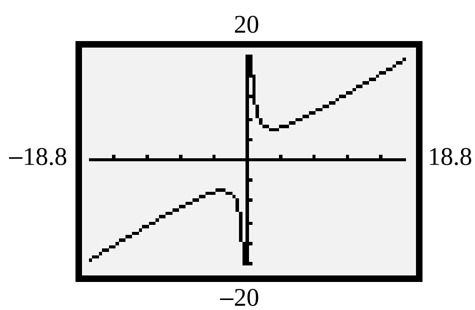


The zeros are approximately −2.1 and 2.1.

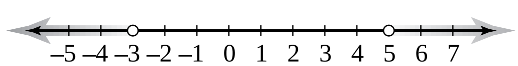
12. a) i)



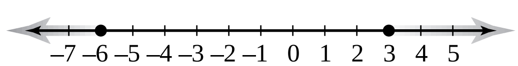
ii)



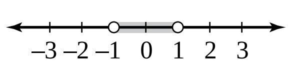
15. a)



b)



c)



17. Explanations may vary. For part a:

$\frac{2-4x}{x-3} \geq x + 4$

This inequality is not defined when $x = 3$. Hence, this solution is valid only when $x \neq 3$.

Selected Solutions — Chapter 8

Case 1: Let $x > 3$. Then $x - 3 > 0$.

Multiply each side by $x - 3$, leaving the inequality sign unchanged.

$$2 - 4x \geq x^2 + x - 12$$
$$0 \geq x^2 + 5x - 14$$
$$(x + 7)(x - 2) \leq 0$$

This occurs when $-7 \leq x \leq 2$. But since this part of the solution is valid only for $x > 3$, we discard this part.

Case 2: Let $x < 3$. Then $x - 3 < 0$.

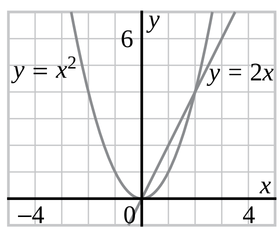
Multiply each side by $x - 3$. Since $x - 3 < 0$, reverse the inequality sign.

$$2 - 4x \leq x^2 + x - 12$$
$$0 \leq x^2 + 5x - 14$$
$$(x + 7)(x - 2) \geq 0$$

This occurs when $x \leq -7$ and $x \geq 2$. But since this part of the solution is valid only when $x < 3$, the solution of the given inequality in this case is $x \leq -7$ and $2 \leq x < 3$.

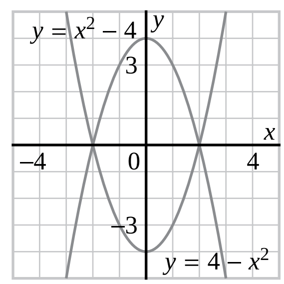
Combining the results of Case 1 and Case 2, the solution of the given inequality is $x \leq -7$ and $2 \leq x < 3$.

19. a)



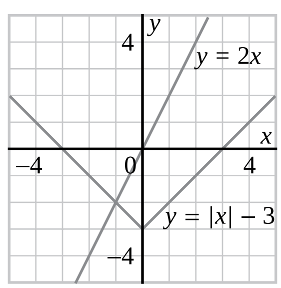
The solution of the system is (0, 0) and (2, 4).

b)



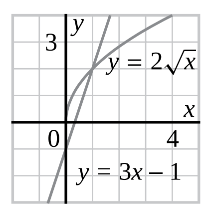
The solution of the system is (–2, 0) and (2, 0).

c)



The solution of the system is (–1, –2).

d)

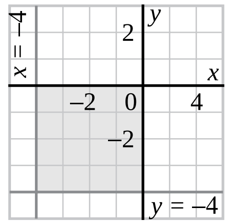


The solution of the system is (1, 2).

20. Explanations may vary. For part c:
I drew the line $y = 2x$ after finding intercept (0, 0) and point (1, 2) on the line. I drew the line $y = |x| - 3$ after translating the graph of $y = |x|$ 3 units down. The two graphs intersect at (–1, –2), which is the solution of the system.

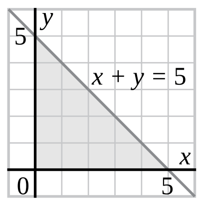
Selected Solutions — Chapter 8

24.



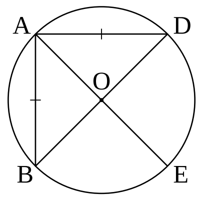
The region is a square. The length of each side of the square is 4 units. Therefore, the area of the region is (4 × 4) square units or 16 square units.

b)



The region is a right triangle. The height of the triangle is 5 units and its base length is 5 units. Therefore, the area of the region is $\left(\frac{1}{2} \times 5 \times 5\right)$ square units or 12.5 square units.

28. Join OB and OD.



Since radii are equal, OB = OA = OD
In △ABO and △ADO
AB = AD
OB = OD
AO is common.
Therefore, △ABO ≅ △ADO (SSS)
Since the triangles are congruent, corresponding angles are equal,
∠BAO = ∠DAO
Since AOE is a straight line, ∠BAE = ∠DAE