Selected Solutions — Chapter 4

Modelling the Time for a Diver to Reach the Water, page 230

The formula that models her height does not take air resistance into account, but air resistance will certainly act on the diver. Thus, her actual time to reach the water may be slightly more than predicted by the formula.

The negative root represents a time before Annie leaves the diving board, when the formula is not valid. Thus, the negative root is not relevant for this situation. For other situations, the negative root may be relevant. For example, imagine that a person is projected upward from the ground at water level, at just the right location, at just the right speed, in just the right direction, and at just the right time so that for $t \geq 0$ the person's motion follows the same path as Annie's. When does the other person have to be projected for this to happen? The negative root gives us the answer: if the other person is projected upwards at $t = -0.29s$, from just the right location at ground level, at just the right speed, and in just the right direction, then after $t = 0$ the two people follow the same path.

4.1 Exercises, page 231

2. b) Explanations may vary. For part i:

I used the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and substituted for a, b, and c the values 2, –5, and 2 respectively. Then I simplified the expression $\frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(2)}}{2(2)}$ to get $\frac{5 \pm 3}{4}$. The provides two roots: 2 and $\frac{1}{2}$.

11. a) The trinomial $ax^2 + bx + c$ can be factored if the quadratic equation $ax^2 + bx + c = 0$ has rational roots. This will be true if the radical part of the quadratic formula is a rational number. Thus, the trinomial can be factored if $\sqrt{b^2 - 4ac}$ is a rational number.

14. Since △ABM and △ADN have the same area, it follows that BM = DN. Let x = BM = DN. Then CM = CN = $6 - x$. The area of △CMN is $\frac{1}{2}(6 - x)^2$, and the area of each other triangle is $\frac{1}{2}(6)x = 3x$. If all the triangles are to have the same area, then

$\frac{1}{2}(6 - x)^2 = 3x$

Simplify this equation and solve for x.

$36 - 12x + x^2 = 6x$

$x^2 - 18x + 36 = 0$

Selected Solutions — Chapter 4

Use the quadratic formula to get:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-18) \pm \sqrt{18^2 - 4(1)(36)}}{2(1)}$$
$$= \frac{18 \pm \sqrt{180}}{2}$$
$$= \frac{18 \pm \sqrt{36 \times 5}}{2}$$
$$= \frac{18 \pm 6\sqrt{5}}{2}$$
$$x = 9 \pm 3\sqrt{5}$$

Thus, there are two potential solutions: $x = 9 + 3\sqrt{5}$ and $x = 9 - 3\sqrt{5}$. Now x is less than 6, so the solution with the positive radical is not relevant. Thus, the solution is

$$x = 9 - 3\sqrt{5}$$
$$\doteq 2.29$$

Thus, BM = DN $\doteq$ 2.29 cm.

15. a) i) $x^2 + x - 1 = 0$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{-1 \pm \sqrt{5}}{2}$$

The roots are $\frac{-1 - \sqrt{5}}{2}$ and $\frac{-1 + \sqrt{5}}{2}$.

ii) $x^2 + x - 2 = 0$

$$= \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2(1)}$$
$$= \frac{-1 \pm \sqrt{9}}{2}$$
$$= -2 \text{ or } 1$$

The roots are -2 and 1.

iii) $x^2 + x - 3 = 0$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-3)}}{2(1)}$$
$$= \frac{-1 \pm \sqrt{13}}{2}$$

The roots are $\frac{-1 - \sqrt{13}}{2}$ and $\frac{-1 + \sqrt{13}}{2}$.

iv) $x^2 + x - 4 = 0$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-4}}{2(1)}$$
$$= \frac{-1 \pm \sqrt{17}}{2}$$

The roots are $\frac{-1 - \sqrt{17}}{2}$ and $\frac{-1 + \sqrt{17}}{2}$.

b) For integral roots, the equation $x^2 - x - n = 0$ must factor. We look for two numbers that differ by -1.

For example, $(x - 2)(x + 1) = 0$
$(x - 3)(x + 2) = 0$
$(x - 4)(x + 3) = 0$ and so on.

The first three values of n are (2)(1) = 2, (3)(2) = 6, (4)(3) = 12
There is an infinite number of values of n.
Any product of consecutive integers is a value for n.

16. a) i) $x^2 + 4x - 45 = 0$

$(x + 9)(x - 5) = 0$

The roots are –9 and 5. Their sum is –4 and their product is –45.

ii) $4x^2 + 20x + 21 = 0$

$(2x + 7)(2x + 3) = 0$

The roots are $-\frac{7}{2}$ and $-\frac{3}{2}$. Their sum is –5 and their product is $\frac{21}{4}$.

iii) $6x^2 - 29x + 35 = 0$

$(2x - 5)(3x - 7) = 0$

The roots are $\frac{5}{2}$ and $\frac{7}{3}$. Their sum is $\frac{29}{6}$ and their product is $\frac{35}{6}$.

iv) $5x^2 - 6x - 3 = 0$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(5)(-3)}}{2(5)} = \frac{6 \pm \sqrt{96}}{10} = \frac{6 \pm 4\sqrt{6}}{10} = \frac{3 \pm 2\sqrt{6}}{5}$$

The roots are $\frac{3 - 2\sqrt{6}}{5}$ and $\frac{3 + 2\sqrt{6}}{5}$.

Their sum is $\frac{3 - 2\sqrt{6}}{5} + \frac{3 + 2\sqrt{6}}{5} = \frac{6}{5}$ and their product is

$$\left(\frac{3 - 2\sqrt{6}}{5}\right)\left(\frac{3 + 2\sqrt{6}}{5}\right) = \frac{9 - 24}{25} = -\frac{3}{5}.$$

b) In each case, the sum is $\frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$, and the product is $\frac{\text{constant term}}{\text{coefficient of } x}$. Hence, for $ax^2 + bx + c = 0$, the sum of the roots is $-\frac{b}{a}$. The product of the roots is $\frac{c}{a}$.

17. The roots of the equation are $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$.

a)
$$\frac{\frac{-b - \sqrt{b^2 - 4ac}}{2a}}{\frac{-b + \sqrt{b^2 - 4ac}}{2a}} = \frac{2}{3}$$

$$-3b - 3\sqrt{b^2 - 4ac} = -2b + 2\sqrt{b^2 - 4ac}$$

$$-5\sqrt{b^2 - 4ac} = b$$

Square each side.

$$25(b^2 - 4ac) = b^2$$

$$24b^2 - 100ac = 0$$

$$b^2 = \frac{25}{6}ac$$

b) $$\frac{\frac{-b-\sqrt{b^2-4ac}}{2a}}{\frac{-b+\sqrt{b^2-4ac}}{2a}} = \frac{m}{n}$$

$$-nb - n\sqrt{b^2-4ac} = -mb + m\sqrt{b^2-4ac}$$

$$(m-n)b = (m+n)\sqrt{b^2-4ac}$$

Square each side.

$$(m-n)^2b^2 = (m+n)^2(b^2-4ac)$$

$$(m-n)^2b^2 = (m+n)^2b^2 - (m+n)^2(4ac)$$

$$(m+n)^2(4ac) = b^2\left[(m+n)^2 - (m-n)^2\right]$$

$$(m+n)^2(4ac) = b^2(4mn)$$

$$mnb^2 = (m+n)^2ac$$

$$b^2 = \frac{(m+n)^2}{mn}ac$$

18. By guess and check, the only examples where the consecutive integers are in order are $-x^2 + 0x + 1$ (or $x^2 - 0x - 1$) and $-2x^2 - x + 0$ (or $2x^2 + x - 0$). Other examples where the integers are consecutive, but not in order, are: $\pm(x^2 + 3x + 2)$, $\pm(2x^2 + 3x + 1)$, $\pm(-x^2 + x + 0)$, $\pm(x^2 - x + 0)$, $\pm(-x^2 - 2x + 0)$.
To be sure that these are the only possible examples, carry out a case-by-case analysis, starting with expressions such as $kx^2 + (k+1)x + (k+2)$. If this expression can be factored, then the quadratic equation $kx^2 + (k+1)x + (k+2) = 0$ has integer roots, which means that $(k+1)^2 - 4k(k+2)$ must be a perfect square. Now simplify and draw conclusions about k. Continue with other expressions, such as $kx^2 + (k+2)x + (k+1)$, until all possibilities have been exhausted.

Mathematical Modelling: How Can We Model a Spiral?, page 236

1. a) 1 unit

b) $(x-1)$ units

2. a) $$\frac{x}{1} = \frac{1}{x-1}$$

$$x^2 - x = 1$$

$$x^2 - x - 1 = 0$$

b) $$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

The positive root is $x = \frac{1+\sqrt{5}}{2}$

$\doteq 1.618\ 033\ 989$

Selected Solutions — Chapter 4

3. a)

Square	Side length	Side length	Decimal form
1	$2\phi - 3$		
2	$2 - \phi$		
3	$\phi - 1$		
4	1	1	
5	ϕ	ϕ	
6	$\phi + 1$		
7	$2\phi + 1$		

b) The side length of square $(n + 2)$ minus the side length of square $(n + 1)$ is equal to the side length of square n. The way the squares are nested in the diagram accounts for this pattern.

4.

Square	Side length	Side length	Decimal form
1	$2\phi - 3$		0.236 067 977
2	$2 - \phi$		0.381 966 011
3	$\phi - 1$		0.618 033 989
4	1	1	1
5	ϕ	ϕ	1.618 033 989
6	$\phi + 1$		2.618 033 989
7	$2\phi + 1$		4.236 067 977

5. a)

$$\begin{aligned}\phi + 1 &= \frac{1 + \sqrt{5}}{2} + 1\\ &= \frac{3 + \sqrt{5}}{2}\\ \phi^2 &= \left(\frac{1 + \sqrt{5}}{2}\right)^2\\ &= \frac{1 + 5 + 2\sqrt{5}}{4}\\ &= \frac{6 + 2\sqrt{5}}{4}\\ &= \frac{3 + \sqrt{5}}{2}\end{aligned}$$

Thus, $\phi + 1 = \phi^2$

Alternatively, note that ϕ is a solution of the equation $x^2 - x - 1 = 0$. Thus, ϕ satisfies this equation:

$\phi^2 - \phi - 1 = 0$

Rearranging the equation results in

$\phi^2 = \phi + 1$

b)

$$\begin{aligned} 2\phi + 1 &= 1 + \sqrt{5} + 1 \\ &= 2 + \sqrt{5} \\ \phi^3 &= (\phi^2)(\phi) \\ &= \left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right) \\ &= \frac{8+4\sqrt{5}}{4} \\ &= 2+\sqrt{5} \end{aligned}$$

Thus, $2\phi + 1 = \phi^3$

Alternatively, from part a

$\phi^2 = \phi + 1$

Multiply both sides of the previous equation by ϕ.

$\phi^3 = \phi^2 + \phi$

$\phi^3 = (\phi + 1) + \phi$, since $\phi^2 = \phi + 1$

$\phi^3 = 2\phi + 1$

c)

$$\begin{aligned} \phi - 1 &= \frac{1+\sqrt{5}}{2} - 1 \\ &= \frac{\sqrt{5}-1}{2} \\ \phi^{-1} &= \frac{1}{\phi} \\ &= \frac{2}{1+\sqrt{5}} \\ &= \frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} \\ &= \frac{2-2\sqrt{5}}{-4} \\ &= \frac{\sqrt{5}-1}{2} \end{aligned}$$

Thus, $\phi - 1 = \phi^{-1}$

Alternatively, once again start with $\phi^2 = \phi + 1$, but this time divide both sides of the equation by ϕ.

$$\begin{aligned} \phi^2 &= \phi + 1 \\ \frac{\phi^2}{\phi} &= \frac{\phi+1}{\phi} \\ \phi &= 1 + \phi^{-1} \\ \phi - 1 &= \phi^{-1} \end{aligned}$$

d)

Square	Side length	Side length	Decimal form
1	$2\phi - 3$	ϕ^{-3}	0.236 067 977
2	$2 - \phi$	ϕ^{-2}	0.381 966 011
3	$\phi - 1$	ϕ^{-1}	0.618 033 989
4	1	1	1
5	ϕ	ϕ	1.618 033 989
6	$\phi + 1$	ϕ^2	2.618 033 989
7	$2\phi + 1$	ϕ^3	4.236 067 977

e) To get from one side length to the next one multiplies by ϕ. This is therefore a geometric sequence with common ratio ϕ.

6. a) To simplify the ratios, use the results of exercise 5. For example, it has been shown that $\phi^2 = \phi + 1$, so in complicated expressions, ϕ^2 can be replaced by $\phi + 1$, ϕ^{-1} can be replaced by $\phi - 1$, and so on.

For example, in the first row of the table,

$$\begin{aligned}\frac{2\phi - 3}{5 - 3\phi} &= \frac{\phi\,(2 - 3\phi^{-1})}{5 - 3\phi} \\ &= \frac{\phi\,(2 - 3(\phi - 1))}{5 - 3\phi} \\ &= \frac{\phi\,(5 - 3\phi)}{5 - 3\phi} \\ &= \phi\end{aligned}$$

Rectangle	Length, *l*	Width, *w*	*l* : *w* ratio
1	$2\phi - 3$	$5 - 3\phi$	ϕ
2	$\phi - 1$	$2 - \phi$	ϕ
3	1	$\phi - 1$	ϕ
4	ϕ	1	ϕ
5	$\phi + 1$	ϕ	ϕ
6	$2\phi + 1$	$1 + \phi$	ϕ
7	$3\phi + 2$	$2\phi + 1$	ϕ

b) They are all golden rectangles.

7. a) $\frac{1 - \sqrt{5}}{2}$

b) Answers may vary. They are negative reciprocals, as the following calculation shows.

$$\begin{aligned}\left(\frac{1 + \sqrt{5}}{2}\right)\left(\frac{1 - \sqrt{5}}{2}\right) &= \frac{1 - 5}{4} \\ &= \frac{-4}{4} \\ &= -1\end{aligned}$$

The roots also have a sum of 1.

$$\frac{1 + \sqrt{5}}{2} + \frac{1 - \sqrt{5}}{2} = 1$$

Thus, the negative root can be written as $-\phi^{-1}$, or $1 - \phi$.

8. a) Use the right triangle drawn in the figure. It has arms of length 1 and 2 units, so by the Pythagorean theorem, the hypotenuse has length *r*, where

$$\begin{aligned}r &= \sqrt{1^2 + 2^2} \\ &= \sqrt{5}\end{aligned}$$

Thus, the radius of the circle is $\sqrt{5}$ units.

b) Again, use the diagram to see that the length of the rectangle is 1 unit larger than the radius of the circle. Thus, the length of the rectangle is $1 + \sqrt{5}$.

c) The length : width ratio of the rectangle is

$$\frac{1 + \sqrt{5}}{2} = \phi$$

d) This rectangle is also a golden rectangle, since its length : width ratio is ϕ.

9. a) If a number x plus 1 is equal to its square, then
$x + 1 = x^2$
which is equivalent to
$x^2 - x - 1 = 0$
It has already been determined that the positive solution to this equation is the golden ratio, $\frac{1 + \sqrt{5}}{2}$.

b) 1.618 033 989

c) The negative number is the other root of $x^2 - x - 1 = 0$, which is $\frac{1 - \sqrt{5}}{2}$.

10. a) If a number x minus 1 is equal to its reciprocal, then
$x - 1 = x^{-1}$
which is equivalent to
$x^2 - x - 1 = 0$
It has already been determined that the positive solution to this equation is the golden ratio, $\frac{1 + \sqrt{5}}{2}$.

b) 1.618 033 989

c) The negative number is the other root of $x^2 - x - 1 = 0$, which is $\frac{1 - \sqrt{5}}{2}$.

11. a) Answers may vary. In the previous exercises, some of the properties of the golden ratio that have arisen are that it is irrational, and that it satisfies the relations $\phi^2 = \phi + 1$ and $\phi^{-1} = \phi - 1$. The golden ratio has many interesting properties and has connections to other situations. For example, you may wish to look up Fibonacci sequences to learn about the connections between them and the golden ratio.

b) Answers may vary; quadratic formula, rectangles, squares, Pythagorean theorem, ratios, irrational numbers, spirals, … .

12. Answers may vary. The name dates from the times of the ancient Greeks.

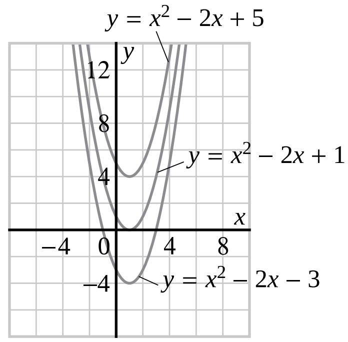
13. Answers may vary. The exact relationships satisfied by the golden ratio (see exercise 11a) can be determined and verified only by using the exact expression for ϕ.

4.2 Exercises, page 244

3. b) Explanations may vary. The nature of the roots is determined by the sign of the discriminant $b^2 - 4ac$. For part i: I substituted $a = 1$, $b = -9$, and $c = 7$ in $b^2 - 4ac$ to get $(-9)^2 - 4(7)$, which simplified to 53.

Selected Solutions — Chapter 4

4. b)



i) $y = x^2 - 2x - 3$ intersects the x-axis twice, hence the equation $x^2 - 2x - 3 = 0$ has two different real roots.

ii) $y = x^2 - 2x + 1$ intersects the x-axis once, hence the equation $x^2 - 2x + 1 = 0$ has two equal real roots.

iii) $y = x^2 - 2x + 5$ does not intersect the x-axis, hence the equation $x^2 - 2x + 5 = 0$ has no real roots.

9. Explanations may vary. For exercise 6a:
For the equation to have two different real roots, $b^2 - 4ac > 0$.
I substituted $a = 1$, $b = k$, and $c = 1$ into $b^2 - 4ac > 0$, to get $k^2 - 4 > 0$, which I wrote as $k^2 > 4$, with the solution $k < -2$ or $k > 2$.

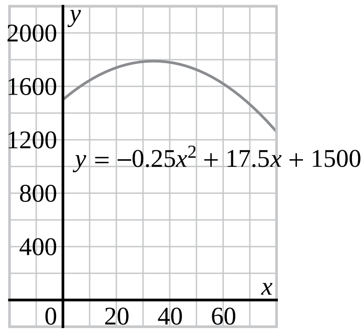
10. b) Since t measures the time that passes after the projectile is fired, only $t \geq 0$ is relevant in this discussion. The projectile will reach a height h if there is a positive value of t that is a solution to the equation $h = 250t - 4.9t^2$.
This equation is equivalent to $0 = -4.9t^2 + 250t - h$.

Thus, the height h is reached provided that the graph of the function $y = -4.9x^2 + 250x - h$ intersects the x-axis at some positive value of x. In part i, the graph does intersect the x-axis for a positive value of x, so there is a time when the height 2750 m is reached. In part ii, the graph does not intersect the x-axis, so we know that the height 4000 m is never reached.

The discriminant in part ii is negative, and that indicates that there are no solutions in part ii. However, the discriminant in part i is positive, but that does not guarantee that there is a time when the height is reached, since it might have happened that both of the solutions had $x \leq 0$, which are not relevant to this problem. Thus, examining the discriminant alone is not conclusive in part i.

11. a) Algebraically, for a profit of \$2000, the equation becomes $-0.25x^2 + 17.5x + 1500 = 2000$. This simplifies to $0.25x^2 - 17.5x + 500 = 0$.
The discriminant $b^2 - 4ac = (-17.5)^2 - 4(0.25)(500)$, which simplifies to -193.75.
Since the discriminant is negative, there is no solution to the equation when the profit is \$2000. There is no value of x that produces a profit of \$2000.
Graphically, the parabola $y = -0.25x^2 + 17.5x + 1500$ does not

reach a value of $y = 2000$. Its maximum value is slightly more than 1800. Once again, there is no value of x that produces a profit of $2000.



12. a) For equal roots $b^2 - 4ac = 0$

Substitute $a = 1$, $b = k$, and $c = 8 - k$.

$k^2 - 4(8 - k) = 0$

$k^2 + 4k - 32 = 0$

$(k + 8)(k - 4) = 0$

$k = -8$ or $k = 4$

For equal roots, $k = -8$ or $k = 4$

b) For real roots $b^2 - 4ac > 0$

Change the equation in part a to an inequality.

$k^2 + 4k - 32 > 0$

$(k + 8)(k - 4) > 0$

Case 1: $k < -8$

$k + 8 < 0$ and $k - 4 < 0$

Hence, $(k + 8)(k - 4) > 0$

Case 2: $-8 < -k < 4$

$k + 8 > 0$ and $k - 4 < 0$

Hence, $(k + 8)(k - 4) < 0$

Case 3: $k > 4$

$k + 8 > 0$ and $k - 4 > 0$

Hence, $(k + 8)(k - 4) > 0$

In Cases 1 and 3, we have the desired results. Thus, for real roots, $k < -8$ or $k > 4$

c) $b^2 - 4ac < 0$

From part b, this occurs when $-8 < k < 4$. Thus, for real roots, $-8 < k < 4$

13. a) i) $x^2 + 50x + 624 = 0$

$$x = \frac{-50 \pm \sqrt{50^2 - 4(624)}}{2}$$

$$= \frac{-50 \pm \sqrt{4}}{2}$$

$$= \frac{-50 \pm 2}{2}$$

$$= -26 \text{ or } -24$$

The roots are -26 and -24.

ii) $x^2 + 50x + 625 = 0$

$$x = \frac{-50 \pm \sqrt{50^2 - 4(625)}}{2} = \frac{-50 \pm 0}{2} = -25$$

The root is -25.

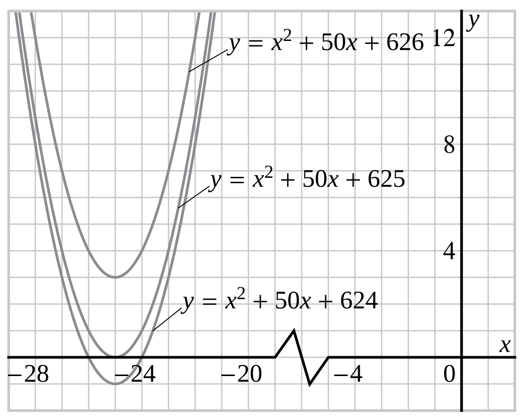
iii) $x^2 + 50x + 626 = 0$

$$x = \frac{-50 \pm \sqrt{50^2 - 4(626)}}{2} = \frac{-50 \pm \sqrt{-4}}{2}$$

There are no real roots.

b) The nature of the roots can change even though there is only a small change in a coefficient.

c) $y = x^2 + 50x + 624$ intersects the x-axis twice.
$y = x^2 + 50x + 625$ intersects the x-axis once.
$y = x^2 + 50x + 626$ does not intersect the x-axis.



14.
$$\frac{1}{x+y} = \frac{1}{x} + \frac{1}{y}$$
$$\frac{1}{x+y} = \frac{y+x}{xy}$$
$$xy = (x+y)^2$$
$$xy = x^2 + 2xy + y^2$$
$$x^2 + xy + y^2 = 0$$

Solve for x in terms of y.

$$x = \frac{-y \pm \sqrt{y^2 - 4y^2}}{2} = \frac{-y \pm \sqrt{-3y^2}}{2}$$

There is no solution unless $y = 0$, in which case also $x = 0$. However, these values do not make any sense in the original equation. Thus, there are no real solutions to the equation $\frac{1}{x+y} = \frac{1}{x} + \frac{1}{y}$.

15. a) Sometimes true. For example, the roots might be $\frac{2 \pm 1}{3}$.

b) Never true. The sum of the roots is $-\frac{b}{a}$, which is rational. So if one root is rational, the other root must be rational.

c) Always true. If the roots are equal, $b^2 - 4ac = 0$. Thus, the roots are both equal to $-\frac{b}{2}$, which is real.

d) Sometimes true. For example, the equation $x^2 + 1 = 0$ has no real roots, and $ac = 1 > 0$.

e) Always true. If $ac < 0$, then $b^2 - 4ac > 0$, which means there are two real roots.

16. a) $x^2 + 4x - 5 = 0$

$(x + 5)(x - 1) = 0$

The roots are -5 and 1.

$x^2 - 5x + 4 = 0$

$(x - 4)(x - 1) = 0$

The roots are 4 and 1.

The common root is 1.

b) It's not clear that there will always be a common root, but if there is one, then let the common root be n. Then

$n^2 + bn + c = 0$ and $n^2 + cn + b = 0$.

Subtract these equations to obtain

$bn + c - cn - b = 0$

$(b - c)n = b - c$

If $b = c$, then the previous equation is satisfied, and the original equations are identical. Suppose that $b \neq c$. Then it's possible to divide both sides of the previous equation by $b - c$ to obtain

$n = 1$.

Thus, the common root, if there is one, is 1. Substitute this value of n into the equations

$n^2 + bn + c = 0$ and $n^2 + cn + b = 0$

to obtain conditions on b and c:

$1 + b + c = 0$ and $1 + c + b = 0$

These conditions are identical: $b + c = -1$

Verify that this condition leads to real solutions. Solve $b + c = -1$ for one of the variables and substitute the resulting expression into the discriminant of each equation.

$b = -c - 1$

$$\begin{aligned} b^2 - 4c &= (-c - 1)^2 - 4c \\ &= c^2 + 2c + 1 - 4c \\ &= c^2 - 2c + 1 \\ &= (c - 1)^2 \geq 0 \end{aligned}$$

and similarly,

$$\begin{aligned} c^2 - 4b &= c^2 - 4(-c - 1) \\ &= c^2 + 4c + 4 \\ &= (c + 2)^2 \geq 0 \end{aligned}$$

Thus, the condition $b + c = -1$ guarantees that the original quadratic equations have a real root in common.

Selected Solutions — Chapter 4

4.3 Exercises, page 254

4. Explanations may vary. For part f:
I used the Remainder Theorem to find the remainder when $x^4 - 3x^3 + 2x^2 - 5x - 1$ is divided by $x - 2$. I substituted $x = 2$ in the polynomial to get $2^4 - 3(2)^3 + 2(2)^2 - 5(2) - 1$. This simplifies to -11, which is the remainder.

9. a) When $f(x)$ is divided by $x - k$, with quotient $q(x)$ and remainder r, the division statement dividend = (divisor)(quotient) + remainder is $f(x) = (x - k)q(x) + r$.

b) The Remainder Theorem states that when a polynomial in x is divided by $x - k$, the remainder is equal to the number obtained by substituting k for x in the polynomial. Substitute $x = k$ in the statement in part a.

$f(k) = (k - k)q(k) + r$

$= 0 + r$

$f(k) = r$

The remainder is r.

4.4 Exercises, page 259

7. Explanations may vary. For exercises 5d:
If $x + 3$ is a factor of $f(x) = -x^4 - 8x^3 - 14x^2 + 8x + 15$, then $f(-3) = 0$. To check, I substituted $x = -3$ to get $f(-3) = -(-3)^4 - 8(-3)^3 - 14(-3)^2 + 8(-3) + 15$, which simplified to 0. Hence, $x + 3$ is a factor.

11. c) To show that $x + 2$ is a factor, I substituted -2 for x in the polynomial $x^3 - 3x^2 - 6x + 8$ to obtain
$(-2)^3 - 3(-2)^2 - 6(-2) + 8 = -8 - 12 + 12 + 8 = 0$.
Thus, by the factor theorem, $x + 2$ is a factor of $x^3 - 3x^2 - 6x + 8$.
To determine the other factors, I wrote the other quadratic factor in the form $x^2 + ax + b$. Then $(x + 2)(x^2 + ax + b) = x^3 - 3x^2 - 6x + 8$.
I expanded the left side and compared coefficients:
$x^3 - (a + 2)x^2 + (b + 2a)x + 2b = x^3 - 3x^2 - 6x + 8$.
Since the corresponding coefficients are equal, $a + 2 = -3$, $b + 2a = -6$, and $2b = 8$.
The first equation results in $a = -5$ and the third equation results in $b = 4$. I substituted both of these values into the second equation.
Then, I wrote the polynomial as
$x^3 - 3x^2 - 6x + 8 = (x + 2)(x^2 - 5x + 4)$. I factored the quadratic expression to obtain $x^3 - 3x^2 - 6x + 8 = (x + 2)(x - 4)(x - 1)$.

Selected Solutions — Chapter 4

16. a)

$$\begin{array}{r|l} & \phantom{2x^3 - {}} x^2 - 3x - 4 \\ \hline 2x - 1 & 2x^3 - 7x^2 - 5x + 4 \\ & \underline{2x^3 - x^2} \\ & -6x^2 - 5x \\ & \underline{-6x^2 + 3x} \\ & -8x + 4 \\ & \underline{-8x + 4} \\ & \phantom{2x^3 - 6x^2 - 8x + {}} 0 \end{array}$$

Since the remainder is zero, $2x - 1$ is a factor.

b) Use the value of x that makes the factor $(2x - 1)$ equal to zero; in other words, use $x = \frac{1}{2}$. To check, substitute this value into the polynomial and verify that the result is zero:

$$\begin{aligned} 2\left(\tfrac{1}{2}\right)^3 - 7\left(\tfrac{1}{2}\right)^2 - 5\left(\tfrac{1}{2}\right) + 4 &= \tfrac{2}{8} - \tfrac{7}{4} - \tfrac{5}{2} + 4 \\ &= 0 \end{aligned}$$

Alternatively, substitute the values of x that make the other factors zero (namely 4 and -1) into the polynomial.

17. Use the extension of the factor theorem that was used in the solution to exercise 16b.

a) Substitute $x = -\frac{1}{3}$ into the polynomial $6x^2 - 10x + 7$ to obtain

$$\begin{aligned} 6\left(-\tfrac{1}{3}\right)^2 - 10\left(-\tfrac{1}{3}\right) + 7 &= \tfrac{6}{9} + \tfrac{10}{3} + 7 \\ &= 11 \end{aligned}$$

Thus, the remainder is 11.

b) Substitute $a = \frac{1}{4}$ into the polynomial $-8a^2 - 2a - 3$ to obtain

$$\begin{aligned} -8\left(\tfrac{1}{4}\right)^2 - 2\left(\tfrac{1}{4}\right)^{-3} &= -\tfrac{8}{16} - \tfrac{2}{4} - 3 \\ &= -4 \end{aligned}$$

Thus, the remainder is -4.

18. Use the extension of the factor theorem that was used in the solution of exercise 16; that is, $2x + 1$ is a factor of the polynomial if the value of x that makes $2x + 1$ equal to zero also makes the polynomial equal to zero.

$$\begin{aligned} 2x + 1 &= 0 \\ x &= -\tfrac{1}{2} \end{aligned}$$

Substitute $-\frac{1}{2}$ for x in the polynomial.

$$\begin{aligned} 2x^3 - x^2 - 13x - 6 &= 2\left(-\tfrac{1}{2}\right)^3 - \left(-\tfrac{1}{2}\right)^2 - 13\left(-\tfrac{1}{2}\right) - 6 \\ &= -\tfrac{2}{8} - \tfrac{1}{4} + \tfrac{13}{2} - 6 \\ &= 0 \end{aligned}$$

Since the result is zero, $2x + 1$ is a factor of the polynomial.

19. The factor theorem can be used: $x - y$ is a factor of $x^n - y^n$ if, when y is substituted for x in $x^n - y^n$, the result is zero. This is true, since $x^n - y^n = 0$ no matter what the value of n is. Thus, $x - y$ is a factor of $x^n - y^n$ for all natural numbers n.

Selected Solutions — Chapter 4

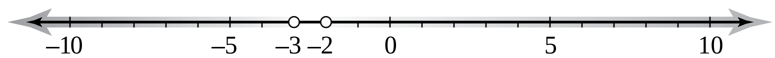
20. The same reasoning is used here as in exercise 19: $x + a$ is a factor of the polynomial $(x + a)^5 + (x + c)^5 + (a - c)^5$ if the polynomial has a value of 0 when $-a$ is substituted for x in the polynomial.

$$\begin{aligned}(-a + a)^5 + (-a + c)^5 + (a - c)^5 &= 0 + (-a + c)^5 + (a - c)^5 \\ &= (-1)^5(a - c)^5 + (a - c)^5 \\ &= -(a - c)^5 + (a - c)^5 \\ &= 0\end{aligned}$$

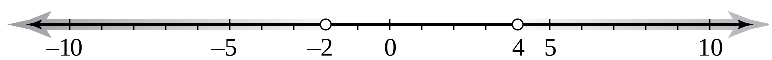
Thus, $x + a$ is a factor of $(x + a)^5 + (x + c)^5 + (a - c)^5$.

4.5 Exercises, page 264

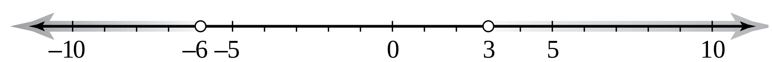
3. a) i)



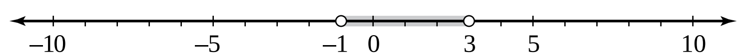
ii)



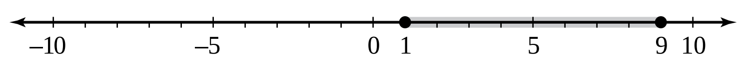
iii)



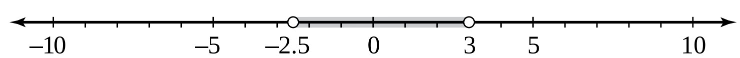
iv)



v)

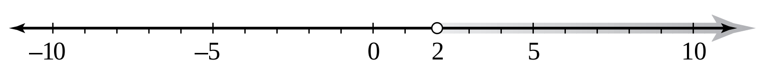


vi)

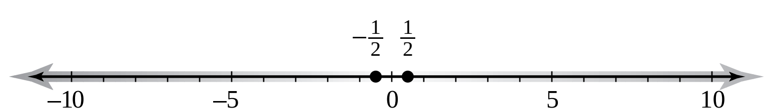


b) Explanations may vary. For part i:
I factored the quadratic expression in $x^2 + 5x + 6 > 0$ to get $(x + 2)(x + 3) > 0$. I visualized the graph of the corresponding quadratic function, $f(x) = x^2 + 5x + 6$. Since the coefficient of x^2 is positive, the graph opens up. It intersects the x-axis at $x = -3$ and $x = -2$. The solution of the inequality $x^2 + 5x + 6 > 0$ consists of the values of x for which the graph of $f(x) = x^2 + 5x + 6$ lies above the x-axis. Hence, the solution is $x < -3$ or $x > -2$.

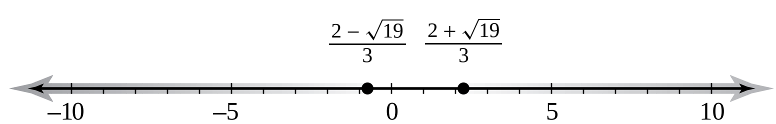
4. a)



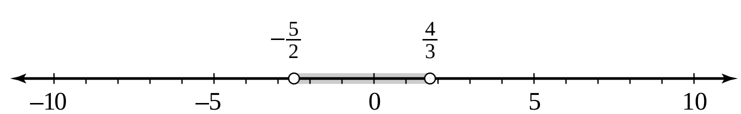
b)



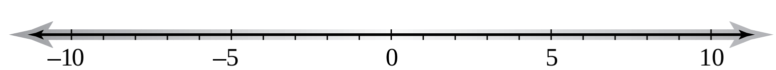
c)



d)

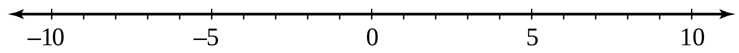


e)

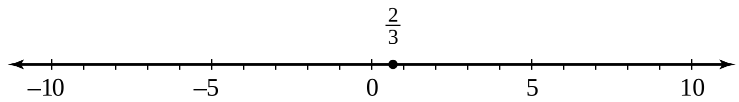


Selected Solutions — Chapter 4

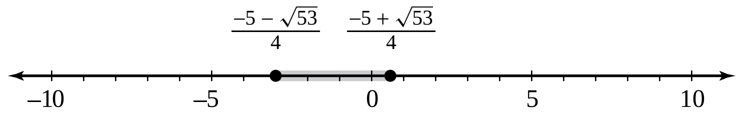
f)



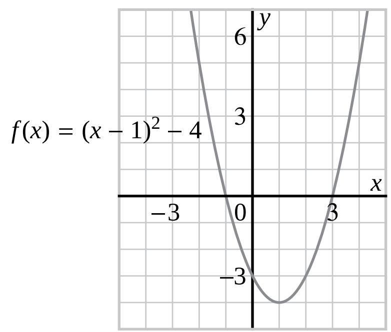
g)



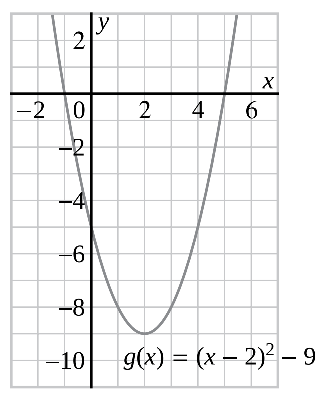
h)



5. a)



6. a)



7. b) Explanations may vary. For part ii: I wrote a quadratic equation with roots –3 and 4:
$(x+3)(x-4) = 0$. Then I expanded to get $x^2 - x - 12 = 0$.
When $-3 < x < 4$, the function $f(x) = x^2 - x - 12$ is below the x-axis. Thus, $x^2 - x - 12 < 0$ is an inequality with the desired solution.

13. a) True. For example, the inequality in exercise 4f.

b) True. For example, the inequality in exercise 4g.

c) False. Parts a and b show that there are quadratic inequalities that do not have infinitely many solutions.

d) False. Every cubic function intersects the horizontal axis at least once.

e) False. Cubic inequalities always have infinitely many solutions, for the same reason as in part d.

f) True, for the reason given in part d.

Selected Solutions — Chapter 4

14. The key to solving this problem is to write the length of the line segment RB in terms of x and y. Triangles $\triangle QAP$ and $\triangle PBR$ are similar. Thus, the ratios of corresponding sides are equal.

To prove that the triangles are similar, label $\angle PQA$ by m, label $\angle QPA$ by n, label $\angle PRB$ by r, and label $\angle RPB$ by s. The sum of the angles in each triangle is 180° and the sum of the three angles at P is also 180°. Thus,

$$m + n + 90^\circ = 180^\circ$$
$$r + s + 90^\circ = 180^\circ$$
$$n + s + 90^\circ = 180^\circ$$

Subtracting the third equation from the first equation results in

$$m - s = 0$$
$$m = s$$

Subtracting the third equation from the second equation results in

$$r - n = 0$$
$$r = n$$

Thus, $\triangle AQP$ is similar to $\triangle BPR$.

Since the triangles are similar, the ratios of corresponding sides are equal. Thus,

$$\frac{AQ}{BP} = \frac{AP}{BR}$$
$$\frac{y}{10 - x} = \frac{x}{BR}$$
$$BR = \frac{10x - x^2}{y}$$

But since the width of the rectangle is 5 cm, $0 < BR < 5$.

$$0 < \frac{10x - x^2}{y} < 5$$

Multiply all terms of the inequality by y.

$$0 < 10x - x^2 < 5y$$
$$y > \frac{10x - x^2}{5}$$

But $y < 5$. Thus,

$$\frac{10x - x^2}{5} < y < 5$$

4.6 Exercises, page 273

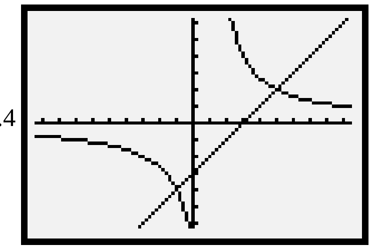
2. b) Explanations may vary. For part i: $\frac{8}{x - 1} = x - 3$. I noted that $x \neq 1$. Then I simplified the equation to remove the fraction. Then I expanded the factors and rearranged the resulting quadratic equation so that it is in standard form. Then I factored the equation.

$$8 = (x - 1)(x - 3)$$
$$8 = x^2 - 4x + 3$$
$$x^2 - 4x - 5 = 0$$
$$(x - 5)(x + 1) = 0$$
$$x = 5 \text{ or } x = -1$$

The roots are 5 and −1. I substituted the roots into the original equation as a check.

Selected Solutions — Chapter 4

3. For exercise 2a, part i:



6. b) Explanations may vary. For part i: $3 + x > \frac{4}{x}$, this inequality is not defined when $x = 0$. Hence, the solution is valid only for $x \neq 0$.

Case 1: Let $x > 0$.

This part of the solution is valid only when $x > 0$.

I multiplied each side by x to get $3x + x^2 > 4$, which I wrote as $x^2 + 3x - 4 > 0$, then factored as $(x + 4)(x - 1) > 0$.

I visualized the graph of the corresponding quadratic function, $f(x) = (x + 4)(x - 1)$. Since the coefficient of x^2 is positive, the graph opens up. It intersects the x-axis at $x = -4$ and $x = 1$. The solution of $(x + 4)(x - 1) > 0$ consists of the values of x for which the graph of $f(x) = (x + 4)(x + 1)$ lies above the x-axis. The solution of $(x + 4)(x - 1) > 0$ is therefore $x < -4$ or $x > 1$. However, since this part of the solution is valid only when $x > 0$, the solution of the given inequality in this case is $x > 1$.

Case 2: Let $x < 0$.

This part of the solution is valid only for $x < 0$.

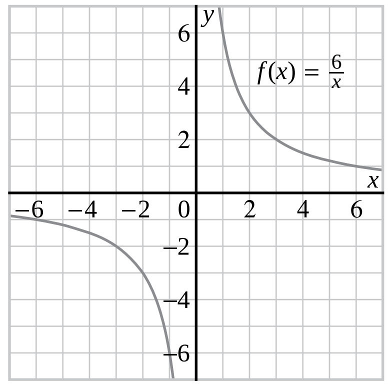
I multiplied each side by x. Since x is negative, I reversed the inequality sign, to get $3x + x^2 < 4$, which I wrote as $x^2 + 3x - 4 < 0$, then factored as $(x + 4)(x - 1) < 0$.

I visualized the graph of the corresponding quadratic function, $f(x) = (x + 4)(x - 1)$. Since the coefficient of x^2 is positive, the graph opens up. It intersects the x-axis at $x = -4$ and $x = 1$. The solution of $(x + 4)(x - 1) < 0$ consists of the values of x for which the graph of $f(x) = (x + 4)(x - 1)$ lies below the x-axis. The solution of $(x + 4)(x - 1) < 0$ is therefore $-4 < x < 1$. However, since this part of the solution is valid only when $x < 0$, the solution of the given inequality in this case is $-4 < x < 0$.

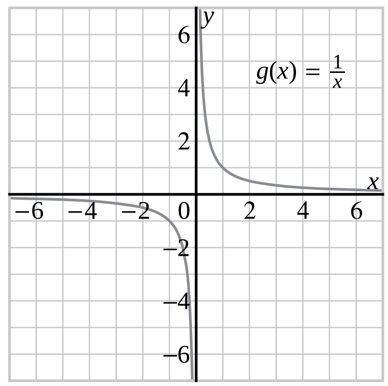
Combining the results of Case 1 and Case 2, the solution of the given inequality is $-4 < x < 0$ or $x > 1$.

Selected Solutions — Chapter 4

8. a)



9. a)



14. From the equation the value of the function is the same if x is replaced by $-x$. Thus, the graph is symmetrical about the y-axis. This fact allows conclusions to be made about negative values of x even though the table shows only positive values of x.

a) From the graph, we can see that $f(x) > 0$ for all values of x.

b) From the table and graph, $f(x) > 0$ for $0 \le x < 2$. However, since the graph is symmetrical about the y-axis, it is also true that $f(x) > 2$ for $-2 < x \le 0$. Putting these two intervals together, we conclude that $f(x) > 2$ for $-2 < x < 2$.

c) From the table and graph, $f(x) > 5$ for $0 \le x < 2$. However, since the graph is symmetrical about the y-axis, it is also true that $f(x) > 5$ for $-1 < x \le 0$. Putting these two intervals together, we conclude that $f(x) > 5$ for $-1 < x < 1$.

d) From the table and graph, $f(x) > 8$ for $0 \le x < 0.5$. However, since the graph is symmetrical about the y-axis, it is also true that $f(x) > 8$ for $-0.5 < x \le 0$. Putting these two intervals together, we conclude that $f(x) > 8$ for $-0.5 < x < 0.5$.

Selected Solutions — Chapter 4

15. a) $\frac{1}{x} < \frac{3}{x+4}$

This inequality is not defined for $x = 0$ and $x = -4$. Hence, the solution is valid only when $x \neq 0$ and $x \neq -4$.

Case 1: Let $x < -4$.

This part of the solution is valid only when $x < -4$.

$\frac{1}{x} < \frac{3}{x+4}$

Multiply each side of the equation by $x(x + 4)$. Since $x < -4$, both x and $x + 4$ are negative, and therefore their product is positive. Therefore, the inequality sign is unchanged.

$x + 4 < 3x$

$2x > 4$

$x > 2$

Since this part of the solution is valid only when $x < -4$, there are no values of x that satisfy the conditions in this case.

Case 2: Let $-4 < x < 0$.

This part of the solution is valid only when $-4 < x < 0$.

$\frac{1}{x} < \frac{3}{x+4}$

Multiply each side of the equation by $x(x + 4)$. Since $-4 < x < 0$, x is negative and $x + 4$ is positive, so their product is negative. Therefore, reverse the inequality sign.

$x + 4 > 3x$

$2x < 4$

$x < 2$

Since this part of the solution is valid only when $-4 < x < 0$, the solution for the given inequality in this case is $-4 < x < 0$.

Case 3: Let $x > 0$.

This part of the solution is valid only when $x > 0$.

$\frac{1}{x} < \frac{3}{x+4}$

Multiply each side of the previous equation by $x(x + 4)$. Since $x > 0$, both x and $x + 4$ are positive. The inequality sign is unchanged.

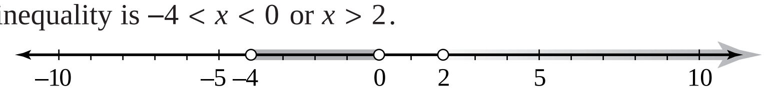
$x + 4 < 3x$

$2x > 4$

$x > 2$

The solution is valid for this case.

Combining the results of Cases 1 to 3, the solution of the given inequality is $-4 < x < 0$ or $x > 2$.



b) $\frac{2}{x-3} < \frac{1}{2x}$

This inequality is not defined for $x = 0$ and $x = 3$. Hence, this solution is valid only when $x \neq 0$ and $x \neq 3$.

Case 1: Let $x < 0$.

This part of the solution is valid only when $x < 0$.

$\frac{2}{x-3} < \frac{1}{2x}$

Multiply each side of the equation by $2x(x-3)$. Since $x < 0$, both $2x$ and $x-3$ are negative, so their product is positive. Therefore, the inequality sign is unchanged.

$4x < x - 3$

$3x < -3$

$x < -1$

This solution is valid for this case.

Case 2: Let $0 < x < 3$.

This part of the solution is valid only when $0 < x < 3$.

$\frac{2}{x-3} < \frac{1}{2x}$

Multiply each side of the equation by $2x(x-3)$. Since $0 < x < 3, x-3$ is negative and $2x$ is positive. Reverse the inequality sign.

$4x > x - 3$

$3x > -3$

$x > -1$

Since this part of the solution is valid only when $0 < x < 3$, the solution for the given inequality in this case is $0 < x < 3$.

Case 3: Let $x > 3$.

This part of the solution is valid only when $x > 3$.

$\frac{2}{x-3} < \frac{1}{2x}$

Multiply each side of the equation by $2x(x-3)$. Since $x > 3$, both $x-3$ and $2x$ are positive. The inequality sign is unchanged.

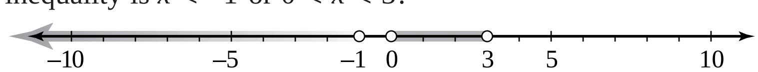
$4x < x - 3$

$3x < -3$

$x < -1$

Since this part of the solution is valid only when $x > 3$, there are no values of x that satisfy the conditions in this case.

Combining the results of Cases 1 to 3, the solution of the given inequality is $x < -1$ or $0 < x < 3$.



c) $\frac{1}{x-1} < \frac{2}{x-2}$

This inequality is not defined for $x = 1$ and $x = 2$. Hence, this solution is valid only when $x \neq 1$ and $x \neq 2$.

Case 1: Let $x < 1$.

This part of the solution is valid only when $x < 1$.

$\frac{1}{x-1} < \frac{2}{x-2}$

Multiply each side of the equation by $(x-1)(x-2)$. Since $x < 1$, both $x - 1$ and $x - 2$ are negative, so their product is positive. Therefore, the inequality sign is unchanged.

$$x - 2 < 2x - 2$$
$$x > 0$$

Since this part of the solution is valid only when $x < 1$, the solution for the given inequality in this case is $0 < x < 1$.

Case 2: Let $1 < x < 2$.

This part of the solution is valid only when $1 < x < 2$.

$\frac{1}{x-1} < \frac{2}{x-2}$

Multiply each side of the equation by $(x-1)(x-2)$. Since $1 < x < 2$, $x - 2$ is negative and $x - 1$ is positive, so their product is negative. Reverse the inequality sign.

$$x - 2 > 2x - 2$$
$$x < 0$$

Since this part of the solution is valid only when $1 < x < 2$, there are no values of x that satisfy the conditions.

Case 3: Let $x > 2$.

This part of the solution is valid only when $x > 2$.

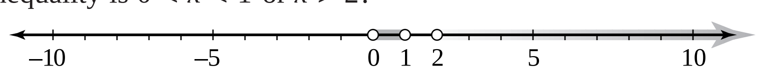
$\frac{1}{x-1} < \frac{2}{x-2}$

Multiply each side of the equation by $(x-1)(x-2)$. Since $x > 2$, both $x - 1$ and $x - 2$ are positive. The inequality sign is unchanged.

$$x - 2 < 2x - 2$$
$$x > 0$$

Since this part of the solution is valid only when $x > 2$, the solution for the given inequality in this case is $x > 2$.

Combining the results of Cases 1 to 3, the solution of the given inequality is $0 < x < 1$ or $x > 2$.



d) $\frac{x}{x-1} \geq \frac{2}{x-2}$

This inequality is not defined for $x = 1$ and $x = 2$. Hence, this solution is valid only when $x \neq 1$ and $x \neq 2$.

Case 1: Let $x < 1$.

This part of the solution is valid only when $x < 1$.

$\frac{x}{x-1} \geq \frac{2}{x-2}$

Multiply each side of the equation by $(x-1)(x-2)$. Since $x < 1$, both $x - 1$ and $x - 2$ are negative, so their product is positive. Therefore, the inequality sign is unchanged.

$$x^2 - 2x \geq 2x - 2$$

$$x^2 - 4x + 2 \geq 0$$

This does not factor, so use the quadratic formula to solve $x^2 - 4x + 2 = 0$, which is

$x = \frac{4 \pm \sqrt{16-8}}{2}$, or $2 \pm \sqrt{2}$

This is true when $x \leq 2 - \sqrt{2}$ or $x \geq 2 + \sqrt{2}$.

Since this part of the solution is valid only when $x < 1$, the solution for the given inequality in this case is $x \leq 2 - \sqrt{2}$.

Case 2: Let $1 < x < 2$.

This part of the solution is valid only when $1 < x < 2$.

$\frac{x}{x-1} \geq \frac{2}{x-2}$

Multiply each side of the equation by $(x-1)(x-2)$. Since $1 < x < 2$, $x - 2$ is negative and $x - 1$ is positive, so their product is negative. Reverse the inequality sign.

$$x^2 - 2x \leq 2x - 2$$

$$x^2 - 4x + 2 \leq 0$$

$$(x - (2 + \sqrt{2}))(x - (2 - \sqrt{2})) \leq 0$$

This is true when $2 - \sqrt{2} \leq x \leq 2 + \sqrt{2}$.

Since this part of the solution is valid only when $1 < x < 2$, the solution for the given inequality in this case is $1 < x < 2$.

Case 3: Let $x > 2$.

This part of the solution is valid only when $x > 2$.

$\frac{x}{x-1} \geq \frac{2}{x-2}$

Multiply each side of the equation by $(x-1)(x-2)$. Since $x > 2$, both $x - 1$ and $x - 2$ are positive. The inequality sign is unchanged.

$$x^2 - 2x \geq 2x - 2$$

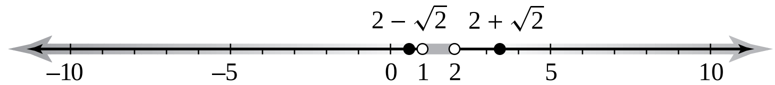
$$x^2 - 4x + 2 \geq 0$$

$$(x - (2 + \sqrt{2}))(x - (2 - \sqrt{2})) \geq 0$$

This is true when $x \leq 2 - \sqrt{2}$ or $x \geq 2 + \sqrt{2}$.

Since this part of the solution is valid only when $x > 2$, the solution for the given inequality in this case is $x \geq 2 + \sqrt{2}$.

Combining the results of Cases 1 to 3, the solution of the given inequality is $x \leq 2 - \sqrt{2}$, $1 < x < 2$, or $x \geq 2 + \sqrt{2}$.



16. a) $\frac{8}{x-3} + \frac{8}{x+1} > -3$

This inequality is not defined for $x = -1$ and $x = 3$. Hence, this solution is valid only when $x \neq -1$ and $x \neq 3$.

Case 1: Let $x < -1$.

This part of the solution is valid only when $x < -1$.

$\frac{8}{x-3} + \frac{8}{x+1} > -3$

Multiply each side of the equation by $(x - 3)(x + 1)$. Since $x < -1$, both $x - 3$ and $x + 1$ are negative, so the product $(x - 3)(x + 1)$ is positive. Thus, the inequality sign is unchanged.

$$8x + 8 + 8x - 24 > -3(x - 3)(x + 1)$$
$$16x - 16 > -3(x^2 - 2x - 3)$$
$$16x - 16 > -3x^2 + 6x + 9$$
$$3x^2 + 10x - 25 > 0$$
$$(3x - 5)(x + 5) > 0$$

This is true when $x > \frac{5}{3}$ or $x < -5$. Since this part of the solution is valid only when $x < -1$, the solution for the given inequality in this case is $x < -5$.

Case 2: Let $-1 < x < 3$.

This part of the solution is valid only when $-1 < x < 3$.

$\frac{8}{x-3} + \frac{8}{x+1} > -3$

Multiply each side of the equation by $(x - 3)(x + 1)$. Since $-1 < x < 3$, $x - 3$ is negative and $x + 1$ is positive, so the product $(x - 3)(x + 1)$ is negative. Reverse the inequality sign.

$$8x + 8 + 8x - 24 < -3(x - 3)(x + 1)$$
$$16x - 16 < -3(x^2 - 2x - 3)$$
$$16x - 16 < -3x^2 + 6x + 9$$
$$3x^2 + 10x - 25 < 0$$
$$(3x - 5)(x + 5) < 0$$

This is true when $-5 < x < \frac{5}{3}$. Since this part of the solution is valid only when $-1 < x < 3$, the solution for the given inequality in this case is $-1 < x < \frac{5}{3}$.

Case 3: Let $x > 3$.

This part of the solution is valid only when $x > 3$.

$\frac{8}{x-3} + \frac{8}{x+1} > -3$

Multiply both sides of the equation by $(x - 3)(x + 1)$. Since $x > 3$, both $x - 3$ and $x + 1$ are positive, so the product $(x-3)(x+1)$ is positive. The inequality sign is unchanged.

$$8x+8+8x-24>-3(x-3)(x+1)$$
$$16x-16>-3(x^2-2x-3)$$
$$16x-16>-3x^2+6x+9$$
$$3x^2+10x-25>0$$
$$(3x-5)(x+5)>0$$

This is true when $x>\frac{5}{3}$ or $x<-5$. Since this part of the solution is valid only when $x>3$, the solution for the given inequality in this case is $x>3$.

Combining the results of Cases 1 to 3, the solution of the given inequality is $x<-5$, $-1<x<\frac{5}{3}$, or $x>3$.

b) $\frac{12}{x-6}+\frac{6}{2x-1}\le -1$

This inequality is not defined for $x=\frac{1}{2}$ and $x=6$. Hence, this solution is valid only when $x\ne\frac{1}{2}$ and $x\ne 6$.

Case 1: Let $x<\frac{1}{2}$.

This part of the solution is valid only when $x<\frac{1}{2}$.

$$\frac{12}{x-6}+\frac{6}{2x-1}\le -1$$

Multiply both sides of the equation by $(x-6)(2x-1)$. Since $x<\frac{1}{2}$, both $x-6$ and $2x-1$ are negative, so the product $(x-6)(2x-1)$ is positive. Thus, the inequality sign is unchanged.

$$24x-12+6x-36\le -(x-6)(2x-1)$$
$$30x-48\le -(2x^2-13x+6)$$
$$30x-48\le -2x^2+13x-6$$
$$2x^2+17x-42\le 0$$
$$(2x+21)(x-2)\le 0$$

This is true when $-\frac{21}{2}\le x\le 2$. Since this part of the solution is valid only when $x<\frac{1}{2}$, the solution for the given inequality in this case is $-\frac{21}{2}\le x<\frac{1}{2}$.

Case 2: Let $\frac{1}{2}<x<6$.

This part of the solution is valid only when $\frac{1}{2}<x<6$.

$$\frac{12}{x-6}+\frac{6}{2x-1}\le -1$$

Multiply both sides of the equation by $(x-6)(2x-1)$. Since $\frac{1}{2}<x<6$, $x-6$ is negative and $2x-1$ is positive, so the product $(x-6)(2x-1)$ is negative. Reverse the inequality sign.

$$24x-12+6x-36\ge -(x-6)(2x-1)$$
$$30x-48\ge -(2x^2-13x+6)$$
$$30x-48\ge -2x^2+13x-6$$
$$2x^2+17x-42\ge 0$$
$$(2x+21)(x-2)\ge 0$$

This is true when $x \le -\frac{21}{2}$ or $x \ge 2$. Since this part of the solution is valid only when $\frac{1}{2} < x < 6$, the solution for the given inequality in this case is $2 \le x < 6$.

Case 3: Let $x > 6$.

This part of the solution is valid only when $x > 6$.

$$\frac{12}{x-6} + \frac{6}{2x-1} \le -1$$

Multiply both sides of the equation by $(x-6)(2x-1)$. Since $x > 6$, both $x - 6$ and $2x - 1$ are positive, so the product $(x-6)(2x-1)$ is also positive. Thus, the inequality sign is unchanged.

$$24x - 12 + 6x - 36 \le -(x-6)(2x-1)$$
$$30x - 48 \le -(2x^2 - 13x + 6)$$
$$30x - 48 \le -2x^2 + 13x - 6$$
$$2x^2 + 17x - 42 \le 0$$
$$(2x+21)(x-2) \le 0$$

This is true when $-\frac{21}{2} \le x \le 2$. Since this part of the solution is valid only when $x > 6$, there are no values of x that satisfy the conditions in this case.

Combining the results of Cases 1 to 3, the solution of the given inequality is $-\frac{21}{2} \le x < \frac{1}{2}$ or $2 \le x < 6$.

c) $\frac{8}{x+6} + \frac{16}{x-8} < x + 3$

This inequality is not defined for $x = -6$ and $x = 8$. Hence, this solution is valid only when $x \ne -6$ and $x \ne 8$.

Case 1: Let $x < -6$.

This part of the solution is valid only when $x < -6$.

$$\frac{8}{x+6} + \frac{16}{x-8} < x + 3$$

Multiply both sides of the equation by $(x+6)(x-8)$. Since $x < -6$, both $x + 6$ and $x - 8$ are negative, so the product $(x+6)(x-8)$ is positive. Thus, the inequality sign is unchanged.

$$8x - 64 + 16x + 96 < (x+3)(x+6)(x-8)$$
$$24x + 32 < x^3 + x^2 - 54x - 144$$
$$x^3 + x^2 - 78x - 176 > 0$$
$$(x+8)(x^2 - 7x - 22) > 0$$

Use the quadratic formula to solve $x^2 - 7x - 22 = 0$, to get $x = \frac{7 \pm \sqrt{49 - 4(-22)}}{2}$, or $x = \frac{7 \pm \sqrt{137}}{2}$.

$$(x+8)\left(x - \left(\frac{7+\sqrt{137}}{2}\right)\right)\left(x - \left(\frac{7-\sqrt{137}}{2}\right)\right) > 0$$

This is true when $-8 < x < \frac{7-\sqrt{137}}{2}$ or $x > \frac{7+\sqrt{137}}{2}$. Since this part of the solution is valid only when $x < -6$, the solution for the given inequality in this case is $-8 < x < -6$.

Selected Solutions — Chapter 4

Case 2: Let $-6 < x < 8$.

This part of the solution is valid only when $-6 < x < 8$.

$\frac{8}{x+6} + \frac{16}{x-8} < x + 3$

Multiply both sides of the equation by $(x+6)(x-8)$. Since $-6 < x < 8$, $x - 8$ is negative and $x + 6$ is positive, so the product $(x+6)(x-8)$ is negative. Reverse the inequality sign.

$$8x - 64 + 16x + 96 > (x+3)(x+6)(x-8)$$
$$24x + 32 > x^3 + x^2 - 54x - 144$$
$$x^3 + x^2 - 78x - 176 < 0$$
$$(x+8)\left(x - \left(\frac{7+\sqrt{137}}{2}\right)\right)\left(x - \left(\frac{7-\sqrt{137}}{2}\right)\right) < 0$$

This is true when $x < -8$ or $\frac{7-\sqrt{137}}{2} < x < \frac{7+\sqrt{137}}{2}$. Since this part of the solution is valid only when $-6 < x < 8$, the solution for the given inequality in this case is $\frac{7-\sqrt{137}}{2} < x < 8$.

Case 3: Let $x > 8$.

This part of the solution is valid only when $x > 8$.

$\frac{8}{x+6} + \frac{16}{x-8} < x + 3$

Multiply both sides of the equation by $(x+6)(x-8)$. Since $x > 8$, both $x + 6$ and $x - 8$ are positive, so the product $(x+6)(x-8)$ is positive. Thus, the inequality sign is unchanged.

$$8x - 64 + 16x + 96 < (x+3)(x+6)(x-8)$$
$$24x + 32 < x^3 + x^2 - 54x - 144$$
$$x^3 + x^2 - 78x - 176 > 0$$
$$(x+8)\left(x - \left(\frac{7+\sqrt{137}}{2}\right)\right)\left(x - \left(\frac{7-\sqrt{137}}{2}\right)\right) > 0$$

This is true when $-8 < x < \frac{7-\sqrt{137}}{2}$ or $x > \frac{7+\sqrt{137}}{2}$. Since this part of the solution is valid only when $x > 8$, the solution for the given inequality in this case is $x > \frac{7+\sqrt{137}}{2}$.

Combining the results of Cases 1 to 3, the solution of the given inequality is $-8 < x < -6$, $\frac{7-\sqrt{137}}{2} < x < 8$, or $x > \frac{7+\sqrt{137}}{2}$.

d) $\frac{4}{x+3} + \frac{4}{2x-4} \geq x - 2$

This inequality is not defined for $x = -3$ and $x = 2$. Hence, this solution is valid only when $x \neq -3$ and $x \neq 2$.

Case 1: Let $x < -3$.

This part of the solution is valid only when $x < -3$.

$$\frac{4}{x+3} + \frac{4}{2x-4} \geq x - 2$$
$$\frac{4}{x+3} + \frac{2}{x-2} \geq x - 2$$

Multiply both sides of the equation by $(x + 3)(x - 2)$. Since $x < -3$, both $x + 3$ and $x - 2$ are negative, so the product $(x + 3)(x - 2)$ is positive. Thus, the inequality sign is unchanged.

$$4x - 8 + 2x + 6 \geq (x - 2)(x - 2)(x + 3)$$
$$6x - 2 \geq x^3 - x^2 - 8x + 12$$
$$x^3 - x^2 - 14x + 14 \leq 0$$
$$(x - 1)(x^2 - 14) \leq 0$$
$$(x - 1)(x - \sqrt{14})(x + \sqrt{14}) \leq 0$$

This is true when $x \leq -\sqrt{14}$ or $1 \leq x \leq \sqrt{14}$. Since this part of the solution is valid only when $x < -3$, the solution for the given inequality in this case is $x \leq -\sqrt{14}$.

Case 2: Let $-3 < x < 2$.

This part of the solution is valid only when $-3 < x < 2$.

$$\frac{4}{x + 3} + \frac{2}{x - 2} \geq x - 2$$

Multiply both sides of the equation by $(x + 3)(x - 2)$. Since $-3 < x < 2$, $x - 2$ is negative and $x + 3$ is positive, so the product $(x + 3)(x - 2)$ is negative. Reverse the inequality sign.

$$4x - 8 + 2x + 6 \leq (x - 2)(x - 2)(x + 3)$$
$$6x - 2 \leq x^3 - x^2 - 8x + 12$$
$$x^3 - x^2 - 14x + 14 \geq 0$$
$$(x - 1)(x - \sqrt{14})(x + \sqrt{14}) \geq 0$$

This is true when $-\sqrt{14} \leq x \leq 1$ or $x \geq \sqrt{14}$. Since this part of the solution is valid only when $-3 < x < 2$, the solution for the given inequality in this case is $-3 < x \leq 1$.

Case 3: Let $x > 2$.

This part of the solution is valid only when $x > 2$.

$$\frac{4}{x + 3} + \frac{2}{x - 2} \geq x - 2$$

Multiply both sides of the previous equation by $(x + 3)(x - 2)$. Since $x > 2$, both $x + 3$ and $x - 2$ are positive, so the product $(x + 3)(x - 2)$ is positive. Thus, the inequality sign is unchanged.

$$4x - 8 + 2x + 6 \leq (x - 2)(x - 2)(x + 3)$$
$$6x - 2 \leq x^3 - x^2 - 8x + 12$$
$$x^3 - x^2 - 14x + 14 \geq 0$$
$$(x - 1)(x - \sqrt{14})(x + \sqrt{14}) \geq 0$$

This is true when $x \leq -\sqrt{14}$ or $1 \leq x \leq \sqrt{14}$. Since this part of the solution is valid only when $x > 2$, the solution for the given inequality in this case is $2 < x \leq \sqrt{14}$.

Combining the results of Cases 1 to 3, the solution of the given inequality is $x \leq -\sqrt{14}$, $-3 < x \leq 1$, or $2 < x \leq \sqrt{14}$.

Selected Solutions — Chapter 4

Problem Solving: The Most Famous Problem in Mathematics, page 276

1. There is an infinite number of examples, such as 3, 4, 5; 6, 8, 10; 12, 16, 20; and so on.

2. a) Answers may vary.
Using $m = 3,\ n = 2$, we obtain $m^2 - n^2 = 5$; $2mn = 12$; $m^2 + n^2 = 13$.
Using $m = 4,\ n = 1$, we obtain $m^2 - n^2 = 15$; $2mn = 8$; $m^2 + n^2 = 17$.
Using $m = 4,\ n = 3$, we obtain $m^2 - n^2 = 7$; $2mn = 24$; $m^2 + n^2 = 25$.

b)
$$\begin{aligned}\text{L.S.} &= (m^2 - n^2)^2 + (2mn)^2\\ &= m^4 - 2m^2n^2 + n^4 + 4m^2n^2\\ &= m^4 + 2m^2n^2 + n^4\\ &= (m^2 + n^2)^2\\ &= \text{R.S.}\end{aligned}$$

3. Answers may vary. Use guess and check.

a) $2^2 + 3^2 + 6^2 = 7^2$

b) $3^3 + 4^3 + 5^3 = 6^3$

c) $2^2 + 9^2 = 6^2 + 7^2$

d) $1^3 + 12^3 = 9^3 + 10^3$

4. Assume there are specific natural numbers a, b, and c that satisfy the equation $x^6 + y^6 = z^6$; that is, $a^6 + b^6 = c^6$.
This can be rewritten as $(a^2)^3 + (b^2)^3 = (c^2)^3$. Thus a^2, b^2, and c^2 are natural numbers that satisfy the equation $x^3 + y^3 = z^3$.
So we have shown that if there are natural numbers that satisfy the equation $x^6 + y^6 = z^6$, then there are natural numbers that satisfy the equation $x^3 + y^3 = z^3$, too. But since the starting assumption of this exercise is that there are no such solutions of the equation $x^3 + y^3 = z^3$, then there can be no such solutions to the equation $x^6 + y^6 = z^6$ either.

5. Answers may vary. Use guess and check.

a) $\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$; $\frac{1}{4} + \frac{1}{12} = \frac{1}{3}$; and so on.

b) For example, for $n = \frac{1}{5}$, write down any addition fact involving natural numbers, such as $2 + 3 = 5$. Now raise each of the three numbers to the fifth power (resulting in 32, 243, and 3125), and you have natural numbers that satisfy the equation $x^{\frac{1}{5}} + y^{\frac{1}{5}} = z^{\frac{1}{5}}$. That is, $32^{\frac{1}{5}} + 243^{\frac{1}{5}} = 3125^{\frac{1}{5}}$, which is identical to $2 + 3 = 5$. The same idea can be used to construct examples for any n that is the reciprocal of a natural number.

4.7 Exercises, page 284

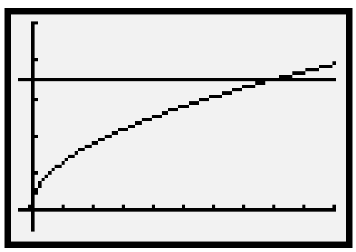
5. b) Explanations may vary. For part i: I noted that the radical is defined only for $x \geq 2$. I isolated the radical on one side of the equation, then squared both sides of the equation. Finally, I solved

the resulting equation for x, and checked that the solution satisfies the restriction on x mentioned above.
$\sqrt{x-2} - 5 = 0$ became $\sqrt{x-2} = 5$, then $x - 2 = 25$, which simplifies to $x = 27$. Then I substituted this value of x into the original equation to verify that it is a valid solution.

6. b) Explanations may vary. For part i: I noted that the radical is defined only for $x \geq 4$. I isolated the radical on one side of the equation, then squared both sides of the equation. The inequality sign is not changed since both sides of the equation are positive before squaring. Finally, I solved the resulting inequality for x, and checked that the solution satisfies the restriction on x mentioned above.
$\sqrt{x-4} - 7 \geq 0$ became $\sqrt{x-4} \geq 7$, then $x - 4 \geq 49$, which simplifies to $x \geq 53$. Since all values of x in the solution satisfy $x \geq 4$, the solution is acceptable.

7. For exercise 3a:

10

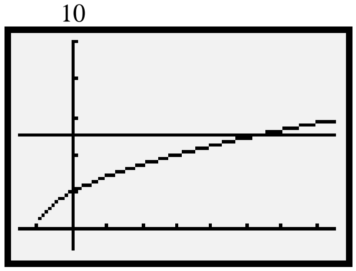


−1 20

−1

For exercise 4a:

10



−3 15

−1

Modelling a Telecommunications Satellite

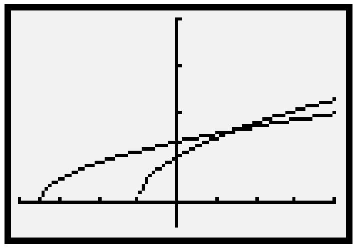
- From exercise 13, the radius of the satellite's orbit is $r \doteq (35\,848 + 6370)$ km $= 42\,218$ km. The distance that the satellite travels in 24 h is the circumference of the orbit, which is
  $2\pi r \doteq 265\,267$ km
- The speed of the satellite is the distance it travels in a day, divided by the time that takes to travel that distance, which is a day:

$$\begin{aligned} \text{speed} &= \frac{\text{distance}}{\text{time}} \\ &\doteq \frac{265\,267}{24} \\ &\doteq 11\,053 \text{ km/h} \end{aligned}$$

Selected Solutions — Chapter 4

14. b) Explanations may vary. For part i: $\sqrt{2x+4} = \sqrt{x+7}$, I noted that this equation is valid only for $x \geq -2$. I squared both sides of the equation and then solved for x to get $2x + 4 = x + 7$, or $x = 3$. Then I verified the solution by substituting in the original equation.

15. For exercise 14a, part i:



18. a) Using the table and graph, when $f(x) > 0$, $x > 0$.

b) Using the table and graph, when $f(x) > 5$, $x > 1$.

c) Using the table and graph, when $f(x) > 10$, $x > 4$.

d) The graph of $y = f(x-1)$ is the image of the graph of $y = f(x)$ after a translation 1 unit right. Visualize the graph of $y = f(x-1)$. When $f(x-1) > 0$, $x > 1$.

e) $f(x-1) - 4 = 5\sqrt{x-1} - 4$

$f(x-1) - 4 > 0$ when $5\sqrt{x-1} - 4 > 0$

$$\begin{aligned} 5\sqrt{x-1} &> 4 \\ \sqrt{x-1} &> 0.8 \\ x - 1 &> 0.64 \\ x &> 1.64 \end{aligned}$$

f) $f(x+1) - 1 = 5\sqrt{x+1} - 1$

$f(x+1) - 1 > 0$ when $5\sqrt{x+1} - 1 > 0$

$$\begin{aligned} 5\sqrt{x+1} &> 1 \\ \sqrt{x+1} &> 0.2 \\ x + 1 &> 0.04 \\ x &> -0.96 \end{aligned}$$

19. a) $\sqrt{x+2} > \frac{x}{x+2}$

For the left side to be defined, $x \geq -2$. For the right side to be defined, $x \neq -2$. Thus, the inequality is true for $x > -2$.

Case 1: $-2 < x \leq 0$

In this interval, the left side of the inequality is greater than 1, and the right side is less than or equal to 0. Thus the inequality is valid over this interval.

Case 2: $x > 0$

For these values of x, the left side is greater than 1 and the right side is less than 1, since the numerator is less than the denominator. Thus, the inequality is valid over this interval, too.

Thus, the solution is $x > -2$.

b) $\sqrt{x+2} > \frac{1}{x+2}$

This has the same restrictions as part a: $x > -2$. Since both sides of the inequality are positive under these circumstances, it is valid to square both sides of the inequality to obtain

$$x + 2 > \frac{1}{(x+2)^2}$$

$$(x+2)^3 > 1$$

Once again, since both sides of the inequality are positive, it is valid to take the cube root of each side to obtain

$$x + 2 > 1$$

Therefore, $x > -1$. Since all values of x in this solution satisfy the condition that $x > -2$, the solution is acceptable.

c) $\sqrt{x+2} > x$

The inequality makes sense only if $x \geq -2$.

Case 1: If $-2 \leq x < 0$, the right side is always negative, and the left side is always positive, so the given inequality is satisfied.

Case 2: Let $x \geq 0$. Then the right side is positive. Since both sides of the inequality are positive, it is valid to square both sides and leave the inequality sign unchanged.

$$x + 2 > x^2$$

$$x^2 - x - 2 < 0$$

$$(x-2)(x+1) < 0$$

This inequality is satisfied when $-1 < x < 2$. Since in this case $x \geq 0$, the solution in this case is $0 \leq x < 2$.

Combining the two cases, the solution is $-2 \leq x < 2$.

d) $\sqrt{x-2} > \frac{1}{x-2}$

The radical is defined for $x \geq 2$, and the right side of the inequality is defined provided that $x \neq 2$. Thus, the inequality is defined for $x > 2$. For such values of x, both sides of the inequality are positive, so it is valid to square both sides and simplify to obtain

$$x - 2 > \frac{1}{(x-2)^2}$$

$$(x-2)^3 > 1$$

$$x - 2 > 1$$

$$x > 3$$

Thus, the solution is $x > 3$.

e) $\sqrt{x+2} + \frac{1}{x+2} > 0$

The radical term is defined for $x \geq -2$, and the rational term is defined provided that $x \neq -2$. Thus, the inequality is defined for $x > -2$. For such values of x, both terms are positive, so the inequality is valid. Thus, the solution is $x > -2$.

Selected Solutions — Chapter 4

f) $\sqrt{x+2} + \frac{x}{x+2} > 0$

As in part e, the inequality makes sense for $x > -2$.

Case 1: $-2 < x < 0$

Rearrange the inequality to obtain

$\sqrt{x+2} > -\frac{x}{x+2}$

Since both sides of the inequality are positive in this case, it is valid to square both sides and rearrange the inequality without changing the inequality sign.

$$\begin{aligned} x+2 &> \frac{x^2}{(x+2)^2} \\ (x+2)^3 &> x^2 \\ x^3 + 6x^2 + 12x + 8 &> x^2 \\ x^3 + 5x^2 + 12x + 8 &> 0 \\ (x+1)(x^2 + 4x + 8) &> 0 \end{aligned}$$

Use the quadratic formula to solve $x^2 + 4x + 8 = 0$; that is, $x = \frac{-4 \pm \sqrt{16-32}}{2}$. Since the discriminant is negative, there is no real solution. Thus, the inequality is valid when $x + 1 > 0$, which means $x > -1$. Since in this case $-2 < x < 0$, the inequality is valid for $-1 < x < 0$.

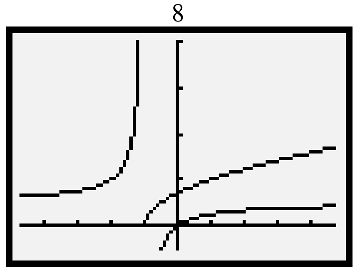
Case 2: $x \geq 0$

In this case, both terms on the left side of the original inequality are positive, so the inequality is valid.

Combining the conclusions of both cases, the solution is $x > -1$.

20. a)

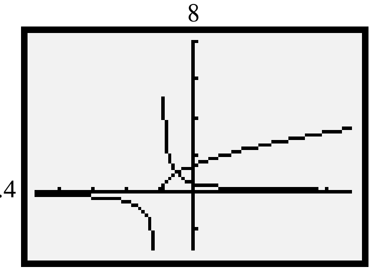
8

–9.4  9.4

–1

b)

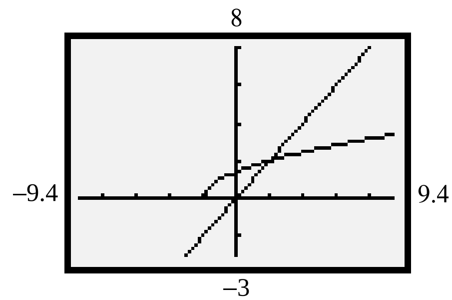
8

–9.4  9.4

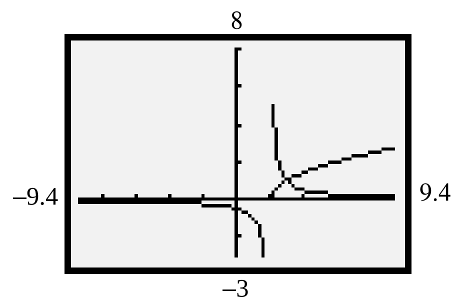
–3

Selected Solutions — Chapter 4

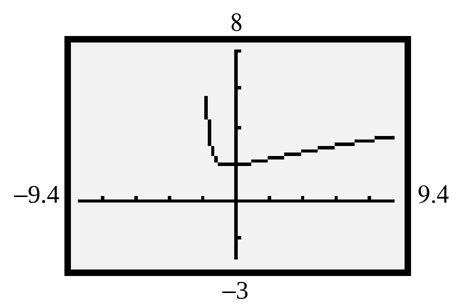
c)



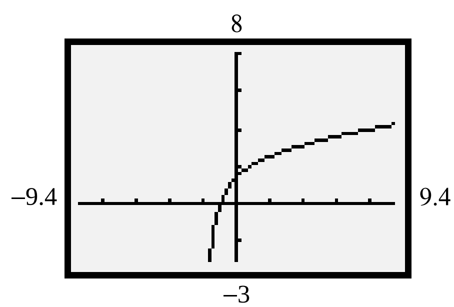
d)



e)



f)



21. a) i) $\sqrt{4x - 12} > \sqrt{2x + 12}$

The left side of the inequality is defined for $x \geq 3$, and the right side is defined for $x \geq -6$. Thus, the inequality is defined for $x \geq 3$. For these values of x, both sides of the inequality are positive, so it is valid to square both sides and leave the inequality sign unchanged.

$$4x - 12 > 2x + 12$$
$$2x > 24$$
$$x > 12$$

The solution for the inequality is $x > 12$. Since all values of x in the solution satisfy $x \geq 3$, the solution is acceptable.

ii) $\sqrt{8x+5} < \sqrt{2x+2}$

The left side of the inequality is defined for $x \geq -\frac{5}{8}$, and the right side is defined for $x \geq -1$. Thus, the inequality is defined for $x \geq -\frac{5}{8}$. For these values of x, both sides of the inequality are positive, so it is valid to square both sides and leave the inequality sign unchanged.

$$8x + 5 < 2x + 2$$
$$6x < -3$$
$$x < -\frac{1}{2}$$

The solution for the given inequality is $x < -\frac{1}{2}$. Since the inequality only makes sense for $x \geq -\frac{5}{8}$, the solution is $-\frac{5}{8} \leq x < -\frac{1}{2}$.

iii) $\sqrt{3x} + \sqrt{x-3} \geq 0$

The first term is defined for $x \geq 0$, and the second term is defined for $x \geq 3$. Thus, the inequality is defined for $x \geq 3$. For these values of x, both terms on the left side of the inequality are positive, so the inequality is certainly valid. Thus, the solution is $x \geq 3$.

iv) $\sqrt{-3x} - \sqrt{x+4} \leq 0$

The first term is defined for $x \leq 0$, and the second term is defined for $x \geq -4$. Thus, the inequality is defined for $-4 \leq x \leq 0$. Rearrange the inequality so that a term that is always positive is on each side; then it is valid to square each side and leave the inequality sign unchanged.

$$\sqrt{-3x} \leq \sqrt{x+4}$$
$$-3x \leq x + 4$$
$$-4 \leq 4x$$
$$-1 \leq x$$

Thus, the solution is $-1 \leq x \leq 0$.

v) $\sqrt{x-1} \geq \sqrt{2x-4}$

The left side of the inequality is defined for $x \geq 1$, and the right side is defined for $x \geq 2$. Thus, the inequality is defined for $x \geq 2$. For these values of x, both sides of the inequality are positive, so it is valid to square both sides and leave the inequality sign unchanged.

$$x - 1 \geq 2x - 4$$
$$x \leq 3$$

Thus, the solution is $2 \leq x \leq 3$.

vi) $\sqrt{x+12} \leq \sqrt{x-4}$

The left side of the inequality is defined for $x \geq -12$, and the right side is defined for $x \geq 4$. Thus, the inequality is defined for $x \geq 4$. For these values of x, both sides of the inequality are positive, so it is valid to square both sides and leave the inequality sign unchanged.

Selected Solutions — Chapter 4

$x + 12 \leq x - 4$

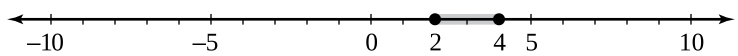
$12 \leq -4$

An impossible statement has been reached, which is a sign that there are no values of x for which the inequality is valid.

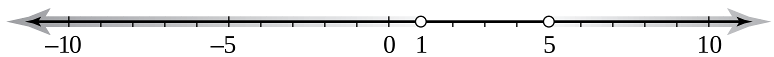
b) Answers may vary. See part a for complete solutions.

4.8 Exercises, page 294

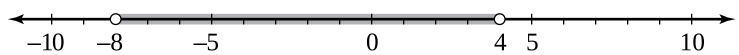
4. a)



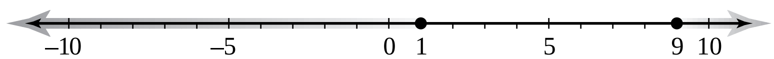
b)



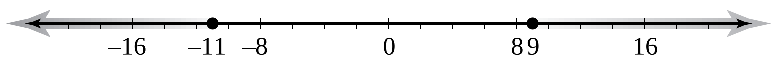
c)



d)

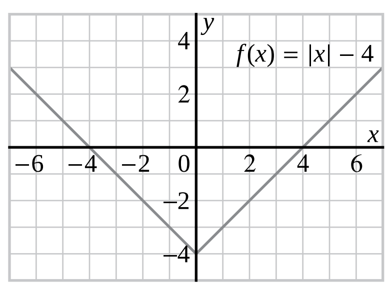


e)

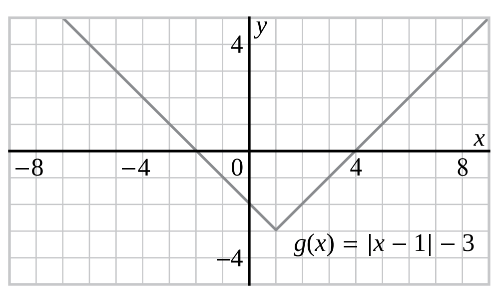


f)

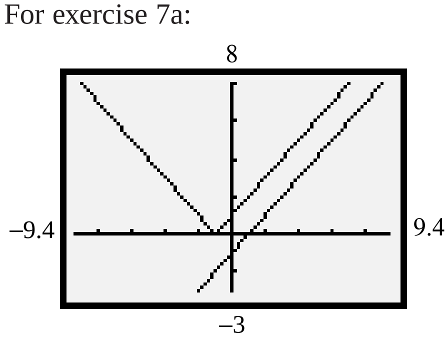
5. a)



6. a)

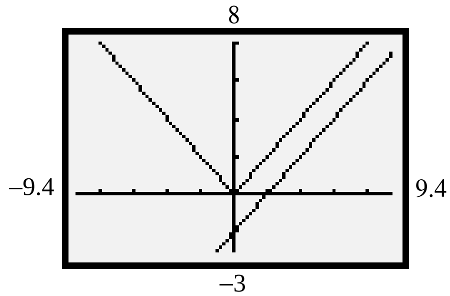


9. For exercise 7a:

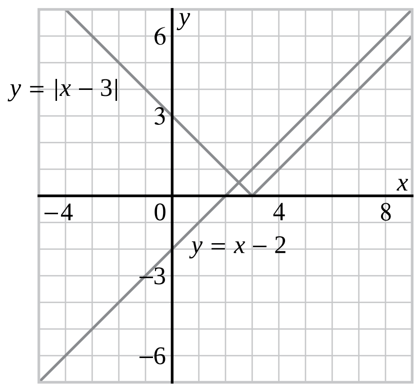


Selected Solutions — Chapter 4

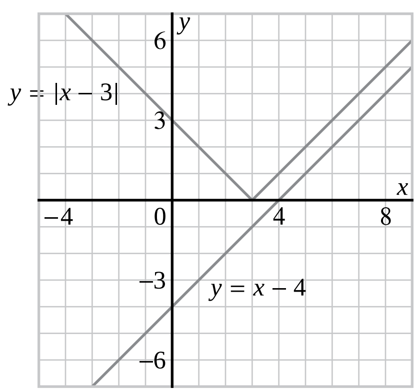
For exercise 8a:



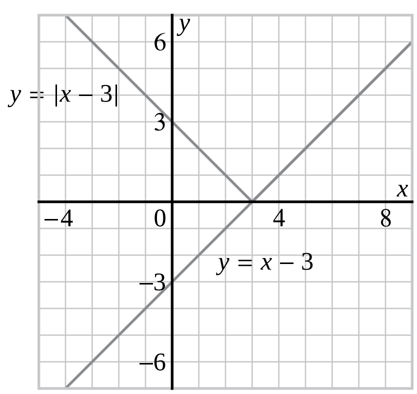
10. a)



b)



c)



12. a) An absolute value cannot be equal to a negative number.

d) The only way this equation could have a solution is if there were a value of x for which both terms on the left side were 0. But there

Selected Solutions — Chapter 4

is no such value, because the first term is 0 only when $x = 2$ and the second term is zero only when $x = -3$.

e) The graphs of $y = x$ and $y = |2x + 1|$ do not intersect.

15. a) $|x - 3| + |x - 8| = 17$

Case 1: $x < 3$

$x - 3 < 0$ and $x - 8 < 0$

$$-x + 3 - x + 8 = 17$$
$$-2x = 6$$
$$x = -3$$

Case 2: $3 \le x < 8$

$x - 3 \ge 0$ and $x - 8 < 0$

$$x - 3 - x + 8 = 17$$
$$5 = 17, \text{ which is impossible}$$

Case 3: $x \ge 8$

$x - 3 > 0$ and $x - 8 \ge 0$

$$x - 3 + x - 8 = 17$$
$$2x = 28$$
$$x = 14$$

The roots are -3 and 14.

b) $|x| + |x - 1| = 5$

Case 1: $x < 0$

$x < 0$ and $x - 1 < 0$

$$-x - x + 1 = 5$$
$$-2x = 4$$
$$x = -2$$

Case 2: $0 \le x < 1$

$x \ge 0$ and $x - 1 < 0$

$$x - x + 1 = 5$$
$$1 = 5, \text{ which is impossible}$$

Case 3: $x \ge 1$

$x > 0$ and $x - 1 \ge 0$

$$x + x - 1 = 5$$
$$2x = 6$$
$$x = 3$$

The roots are -2 and 3.

c) $|2x - 1|$ and $|1 - 2x|$ are identical, and therefore the left side of the equation is always 0, no matter what x is. Hence, there is no solution.

d) $|x - 1| + |x - 3| = 6$

Case 1: $x < 1$

$x - 1 < 0$ and $x - 3 < 0$

$$1 - x + 3 - x = 6$$
$$-2x = 2$$
$$x = -1$$

Selected Solutions — Chapter 4

Case 2: $1 \le x < 3$

$x - 1 \ge 0$ and $x - 3 < 0$

$x - 1 - x + 3 = 6$

$2 = 6$, which is impossible

Case 3: $x \ge 3$

$x - 1 > 0$ and $x - 3 \ge 0$

$x - 1 + x - 3 = 6$

$2x = 10$

$x = 5$

The roots are -1 and 5.

16. a) $|x - 1| + |x - 3| \le 2$

Case 1: $x < 1$

$x - 1 < 0$ and $x - 3 < 0$

$1 - x + 3 - x \le 2$

$-2x \le -2$

$x \ge 1$

This solution is extraneous, because in this case, $x < 1$.

Case 2: $1 \le x \le 3$

$x - 1 \ge 0$ and $x - 3 \le 0$

$x - 1 - x + 3 \le 2$

$2 \le 2$

This is true for all x in this case. Thus, the solution for the given inequality in this case is $1 \le x \le 3$.

Case 3: $x > 3$

$x - 1 > 0$ and $x - 3 > 0$

$x - 1 + x - 3 \le 2$

$2x \le 6$

$x \le 3$

Since $x > 3$ in this case, there is no solution in this case.

Combine the solutions in all three cases to obtain $1 \le x \le 3$.

b) $|x + 2| + |2 - x| < 8$

Case 1: $x < -2$

$x + 2 < 0$ and $2 - x > 0$

$-x - 2 + 2 - x < 8$

$-2x < 8$

$x > -4$

The solution of the given inequality in this case is $-4 < x < -2$.

Case 2: $-2 \le x \le 2$

$x + 2 \ge 0$ and $2 - x \ge 0$

$x + 2 + 2 - x < 8$

$4 < 8$

This is true for all x in this case. Thus, the solution for the given inequality in this case is $-2 \le x \le 2$.

Selected Solutions — Chapter 4

Case 3: $x > 2$

$x + 2 > 0$ and $2 - x < 0$

$x + 2 - 2 + x < 8$

$2x < 8$

$x < 4$

The solution of the given inequality in this case is $2 < x < 4$.

Combine the solutions in all three cases to obtain $-4 < x < 4$.

c) $|x + 1| + |2x - 5| > 5$

Case 1: $x < -1$

$x + 1 < 0$ and $2x - 5 < 0$

$-x - 1 - 2x + 5 > 5$

$-3x > 1$

$x < -\frac{1}{3}$

The solution of the given inequality in this case is $x < -1$.

Case 2: $-1 \le x \le \frac{5}{2}$

$x + 1 \ge 0$ and $2x - 5 \ge 0$

$x + 1 - 2x + 5 > 5$

$-x > -1$

$x < 1$

The solution of the given inequality in this case is $-1 \le x < 1$.

Case 3: $x > \frac{5}{2}$

$x + 1 > 0$ and $2x - 5 > 0$

$x + 1 + 2x - 5 > 5$

$3x > 9$

$x > 3$

The solution of the given inequality in this case is $x > 3$.

Combine the solutions in all three cases to obtain $x < 1$ or $x > 3$.

d) $|x + 2| - |x| \ge 4$

Case 1: $x < -2$

$x + 2 < 0$ and $x < 0$

$-x - 2 + x \ge 4$

$-2 \ge 4$, which is impossible

Case 2: $-2 \le x \le 0$

$x + 2 \ge 0$ and $x \le 0$

$x + 2 + x \ge 4$

$2x \ge 2$

$x \ge 1$

This solution is extraneous since in this case $-2 \le x \le 0$.

Selected Solutions — Chapter 4

Case 3: $x > 0$

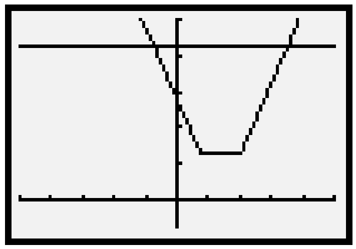
$x + 2 > 0$ and $x > 0$

$x + 2 - x > 4$

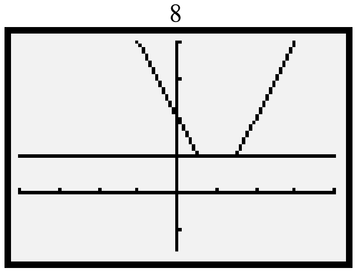
$2 > 4$, which is impossible

There is no solution to the given inequality.

17. For exercise 15a:



For exercise 16a:



18. Answers may vary. The first thing one ought to try is inequalities involving only $|x|$, as they are likely to be the simplest.

a) $|x| < 0$ **b)** $|x| \geq 0$ **c)** $|x| \leq 0$

19. a) $|x + 2| \geq \frac{1}{x+2}$

Note that $x \neq -2$.

Case 1: $x < -2$

If $x < -2$, the right side is negative, so $|x + 2| \geq \frac{1}{x+2}$ for all values of $x < -2$.

Case 2: $x > -2$

$x + 2 \geq \frac{1}{x+2}$

$(x + 2)^2 \geq 1$

$x + 2 \leq -1$ or $x + 2 \geq 1$

$x \leq -3$ or $x \geq -1$

But $x > -2$, so the solution in this case is $x \geq -1$.

Combine the two cases to obtain a solution of $x < -2$ or $x \geq -1$.

b) $|x + 2| \geq \frac{x}{x+2}$

Note that $x \neq 2$.

Case 1: $x < -2$

$x + 2 < 0$

$-(x + 2) \geq \frac{x}{x + 2}$

Multiply each side by $x + 2$, which is negative, so reverse the inequality sign.

$-(x + 2)^2 - x \leq 0$

$-x^2 - 5x - 4 \leq 0$

$x^2 + 5x + 4 \geq 0$

$(x + 1)(x + 4) \geq 0$

This is true when $x \leq -4$ or $x \geq -1$.

But $x < -2$ in this case, so the solution is $x \leq -4$.

Case 2: $x > -2$

$x + 2 > 0$

$x + 2 \geq \frac{x}{x + 2}$

$(x + 2)^2 - x \geq 0$

$x^2 + 3x + 4 \geq 0$

Use the quadratic formula to solve $x^2 + 3x + 4 = 0$ to get $x = \frac{-3 \pm \sqrt{9 - 4(4)}}{2}$. Since there are no real solutions, the curve $y = x^2 + 3x + 4$ opens up and is above the x-axis. So the inequality is true for all values of x. But $x > -2$, so the solution in this case is $x > -2$.

Combine the two cases to obtain a solution of $x \leq -4$ or $x > -2$.

4 Review, page 296

2. Explanations may vary. For part a: I expanded the factors, then collected all the terms on one side of the equation so that the resulting quadratic equation will be in standard form.

$(2x + 1)(3x - 2) = (4x - 7)(x - 3)$

$6x^2 - x - 2 = 4x^2 - 19x + 21$

$2x^2 + 18x - 23 = 0$

I used the quadratic formula to solve the equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-18 \pm \sqrt{324 - 4(2)(-23)}}{2(2)}$$

$$= \frac{-18 \pm \sqrt{508}}{4}$$

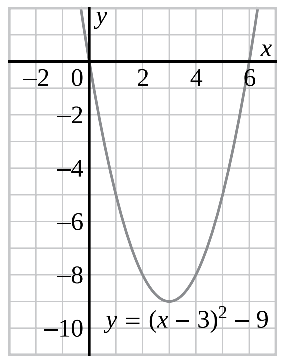
$$\doteq 1.13 \text{ or } -10.13$$

The roots are 1.13 and -10.13.

6. Explanations may vary. For part a: To find the remainder when the polynomial is divided by $x - 1$, I substituted $x = 1$ in the polynomial to get $x^3 + x^2 - x - 1 = 1 + 1 - 1 - 1$, or 0. The remainder is 0.

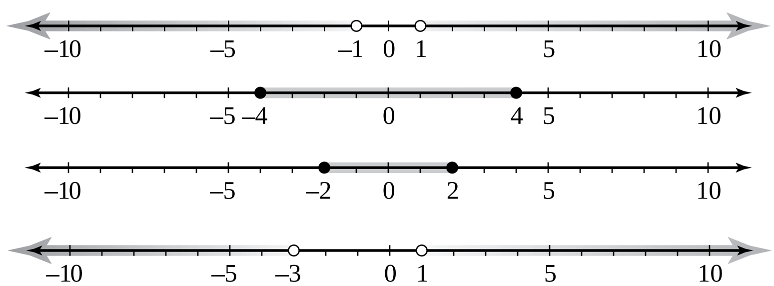
Selected Solutions — Chapter 4

8. a)



12. Answers may vary. For part a: $\sqrt{2x + 5} = 5$, I squared each side to get $2x + 5 = 25$, which simplified to $2x = 20$ or $x = 10$. The root is 10. Then I checked the solution by substituting $x = 10$ into the original equation.

17. a) **b)** **c)** **d)**



19. Explanations may vary. For part a: $|x| \geq 2x + 1$; I considered two cases.

Case 1: $x \geq 0$, then $|x| = x$ and I wrote the inequality as $x > 2x + 1$, which simplified to $x \leq -1$. I rejected this solution because it is not among the possible values for x, since in this case $x \geq 0$.

Case 2: $x < 0$, then $|x| = -x$ and I wrote the inequality as $-x \geq 2x + 1$, which simplified to $-1 \geq 3x$, or $x \leq -\frac{1}{3}$. This is among the possible values for x in this case.

Thus, the solution is $x \leq -\frac{1}{3}$.

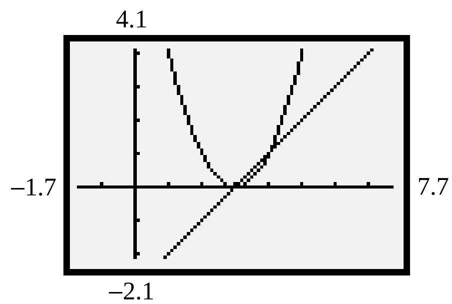
20. a) $|3x + 1| = -2$; this equation has no solution since an absolute value cannot be equal to a negative number.

c) $2|x| = 4|x| + 3$; rearranging this equation leads to $2|x| = -3$. This equation has no solution since an absolute value cannot be equal to a negative number.

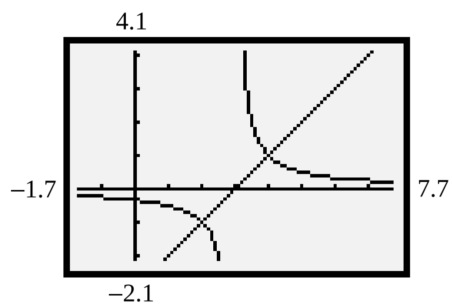
f) Since absolute values cannot be negative, for this equation to have a solution, there must be a single value of x for which $|x - 3| = 0$ and $|3x - 2| = 0$. Since there is no such value of x, the original equation has no solution.

22. b) Answers may vary. Parts i and iii have the same solution. The solution to part iv is an entire ray of the x-axis, whereas the solutions to the other parts are just two points. The equations in the first three parts all reduce to quadratic equations, so no more than two solutions are expected. In part iv, consider the plane divided into two parts, the right-hand part for which $x \geq 3$, and the left-hand part for which $x < 3$. On each part of the plane, the equation reduces to the intersection of two lines; on the left-hand side the lines do not intersect, and on the right-hand side the two lines are identical, so they intersect at every point.

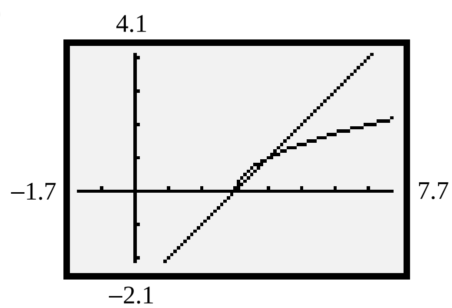
i)



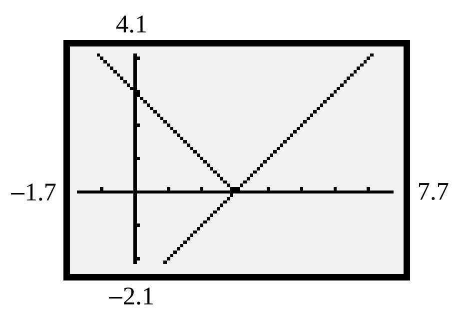
ii)



iii)



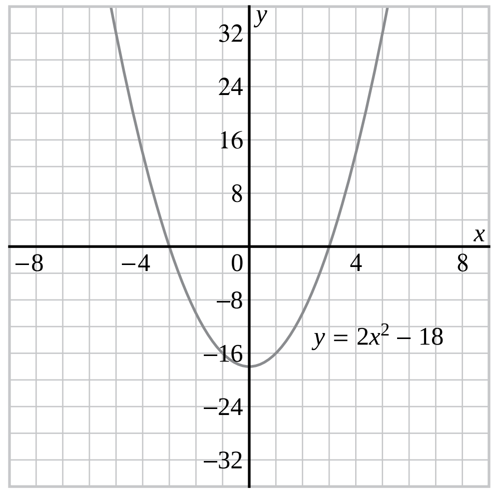
iv)



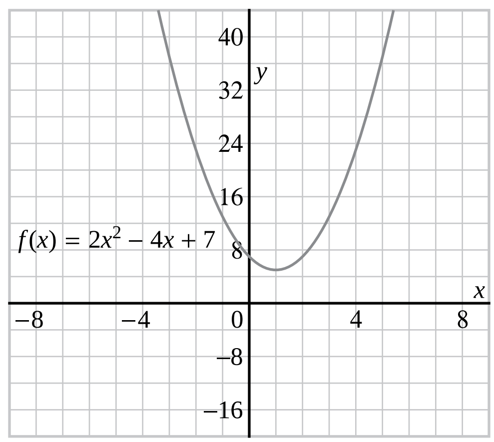
Selected Solutions — Chapter 4

Cumulative Review, page 298

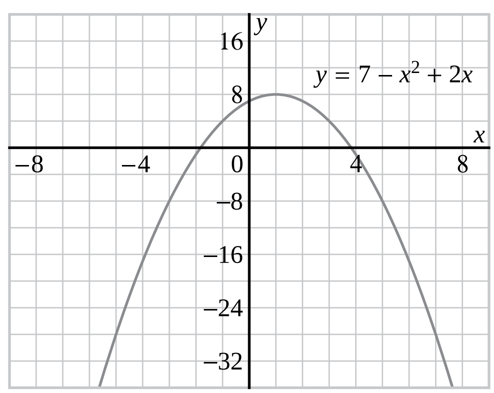
7. a)



b)



c)

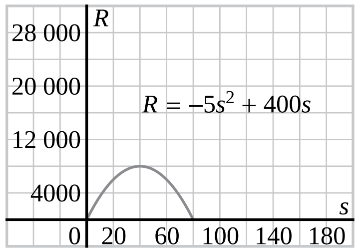


8. Explanations may vary. For part a: Since the coefficient of x^2 is positive, the graph is a parabola opening upwards. Next, I determined the intercepts. The y-intercept is -18, and the x-intercepts are ± 3. This information is sufficient to allow me to draw a rough sketch of the graph; I used the method of differences to determine a few more points.

Selected Solutions — Chapter 4

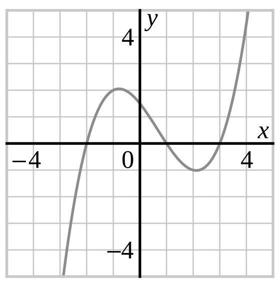
10. Explanations may vary. For part a: I chose factors that are zero at the indicated roots; that is, $(x - 2)$ and $(x - (-6))$. Thus, any function that satisfies the stated condition has the form $y = a(x - 2)(x + 6)$. I chose $a = 1$ to get the function $y = (x - 2)(x + 6)$, which I then wrote as $y = x^2 + 4x - 12$.

11. a)

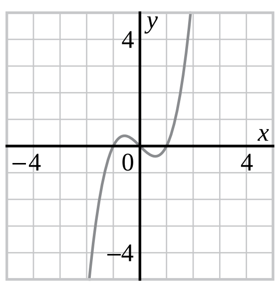


15. Explanations may vary. For part b: Since the function has a variable in the denominator, it is a rational function, not a polynomial function.

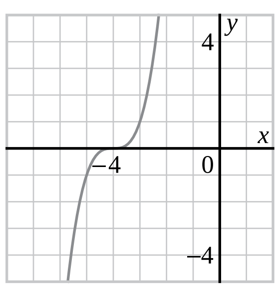
16. a)



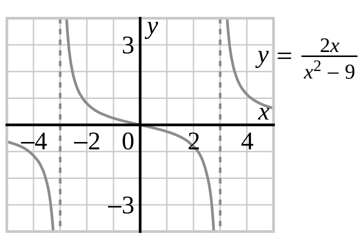
b)



c)

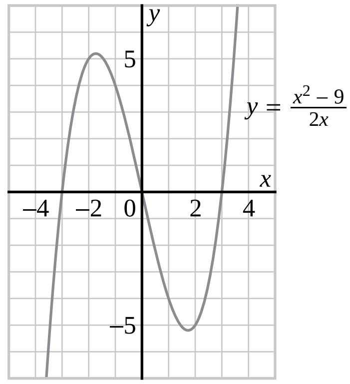


21. a)



Selected Solutions — Chapter 4

b)



23. The grappling iron will reach the ledge if the maximum value of the function is greater than or equal to 7.5 m. Complete the square to get $h = -4.9\left(t^2 - \frac{11}{4.9}t + \left(\frac{11}{9.8}\right)^2 - \left(\frac{11}{9.8}\right)^2\right) + 1.5$, which reduces to $h \doteq -4.9\left(t - \frac{11}{9.8}\right)^2 + 7.67$. The maximum value is 7.67, which is greater than 7.5.