Selected Solutions — Chapter 6

Modelling Earth, page 326

1. a) The polar radius is 6365 km – 8 km = 6357 km.

$$\begin{aligned} A &= 4\pi r^2 \\ &= 4 \times \pi \times 6357^2 \\ &\doteq 5.08 \times 10^8 \text{ km}^2 \end{aligned}$$

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3} \times \pi \times 6357^3 \\ &\doteq 1.08 \times 10^{12} \text{ km}^3 \end{aligned}$$

b) The equatorial radius is 6365 km + 13 km = 6378 km.

$$\begin{aligned} A &= 4\pi r^2 \\ &= 4 \times \pi \times 6378^2 \\ &\doteq 5.11 \times 10^8 \text{ km}^2 \end{aligned}$$

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3} \times \pi \times 6378^3 \\ &\doteq 1.09 \times 10^{12} \text{ km}^3 \end{aligned}$$

2. The actual surface area and volume would be closer to the results obtained with the polar radius, because this radius is closer to the mean radius.

6.1 Exercises, page 328

7. Answers may vary. For part a: The surface area is 2367 mm^2.
I substituted this value for A in the formula $A = 4\pi r^2$, then I solved the equation for r.

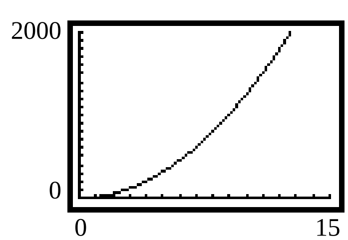
$$\begin{aligned} 4\pi r^2 &= 2367 \\ r^2 &= \frac{2367}{4 \times \pi} \\ r &= \sqrt{\frac{2367}{4 \times \pi}} \\ &\doteq 13.7244 \end{aligned}$$

I multiplied this value for r by 2 to get the diameter, d.

$$\begin{aligned} d &= 2r \\ &\doteq 2 \times 13.7244 \\ &= 27.4488 \end{aligned}$$

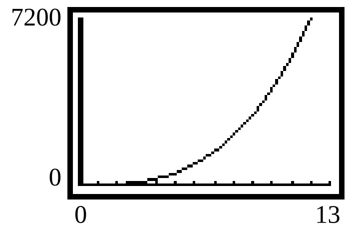
The diameter is about 27.4 mm.

12. a) $A = 4\pi r$

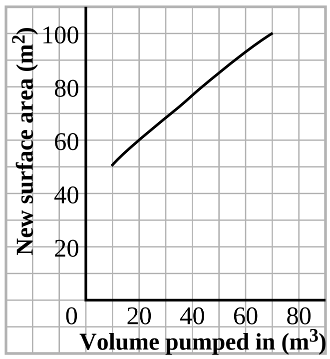


Selected Solutions — Chapter 6

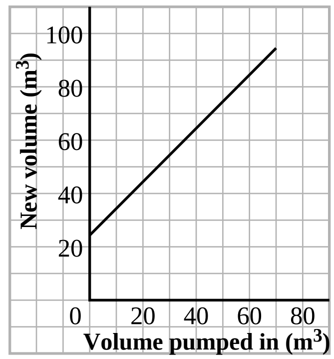
13. a) $V = \frac{4}{3}\pi r^3$



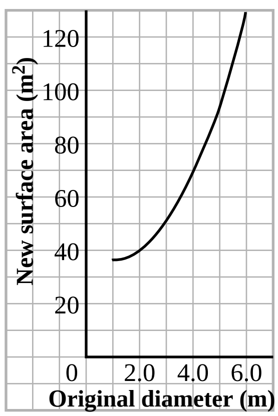
14. b) **Surface area against volume pumped in**



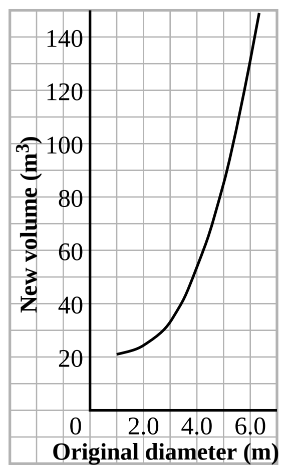
c) **Volume against volume pumped in**



15. b) **Surface area against original diameter**



c) **Volume against original diameter**



16. The volume of a sphere with radius r is $\frac{4}{3}\pi r^3$. The volume of a cube with edges of length r is r^3. $\frac{4}{3}\pi$ is approximately equal to 4.19, which is greater than 1. Therefore, $\frac{4}{3}\pi r^3 > r^3$, and the volume of a sphere with radius r is greater than the volume of a cube with edges of length r.

20. Answers may vary. For part a: The surface area of a sphere is given by $A = 4\pi r^2$

The radius r is one-half the diameter d.

$$r = \frac{d}{2}$$

Selected Solutions — Chapter 6

Substitute for r in the formula for A.

$$A = 4\pi(\tfrac{d}{2})^2$$
$$= 4\pi(\tfrac{d^2}{4})$$
$$= \pi d^2$$

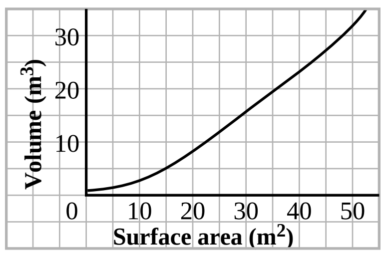
The volume of a sphere is given by

$$V = \tfrac{4}{3}\pi r^3$$

Substitute for r.

$$V = \tfrac{4}{3}\pi(\tfrac{d}{2})^3$$
$$= \tfrac{4}{3}\pi(\tfrac{d^3}{8})$$
$$= \frac{\pi d^3}{6}$$

21. a) **Volume against surace area for a sphere**



c) Larger animals are better able to maintain their body heat in cold climates because their value of $\frac{V}{A}$ is larger than that of smaller animals.

22. a) The volume of air in the hemisphere is the same as the volume in the sphere, because the sphere has changed shape to a hemisphere, and has not lost any air.

b) For the hemisphere to have the same volume as the sphere, it must have a greater radius. The volume of a sphere is given by $V_s = \frac{4}{3}\pi R^3$.

The volume of a hemisphere is given by $V_h = \frac{2}{3}\pi r^3$.

If $V_s = V_h$, then $r > R$

c) The volume of a sphere of radius R is given by $V = \frac{4}{3}\pi R^3$. The volume of a hemisphere of radius r is $\frac{1}{2} \times \frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3$.

The volumes of the sphere and the hemisphere are equal.

$$\tfrac{2}{3}\pi r^3 = \tfrac{4}{3}\pi R^3$$

Divide each term by $\frac{2\pi}{3}$.

$$r^3 = 2R^3$$
$$\frac{r^3}{R^3} = 2$$
$$\frac{r}{R} = \sqrt[3]{2}$$

The ratio radius of hemisphere : radius of sphere is

$\sqrt[3]{2} : 1$.

d) The surface area of a sphere of radius R is given by $A = 4\pi R^2$. The curved surface area of a hemisphere of radius r is

$\frac{1}{2} \times 4\pi r^2 = 2\pi r^2$.

The flat surface of the hemisphere has area πr^2. The total surface area of the hemisphere is $2\pi r^2 + \pi r^2 = 3\pi r^2$.

$$\frac{\text{Surface area of hemisphere}}{\text{Surface area of sphere}} = \frac{3\pi r^2}{4\pi R^2} = \frac{3r^2}{4R^2}$$

Substitute $\sqrt[3]{2}$ for $\frac{r}{R}$.

$$\begin{aligned}\frac{\text{Surface area of hemisphere}}{\text{Surface area of sphere}} &= \frac{3}{4}(\sqrt[3]{2})^2 \\ &= 0.75\sqrt[3]{4}\end{aligned}$$

The ratio surface area of hemisphere : surface area of sphere is

$0.75\sqrt[3]{4} : 1$.

23. a) The volume of space that can be studied by the Space Telescope has a radius of 14 billion light-years, which is 7 times the radius of the volume of space that can be studied by ground observatories. Since the Space Telescope radius is 7 times the ground observatory radius, the Space Telescope volume is $7^3 = 343$ times the ground observatory volume.

b) The volume of space that can be studied by ground observatories has a radius of 2 billion light-years. The volume of space that can be studied by the unaided eye has a radius of 600 000 light-years. Divide these volumes: $\frac{2\,000\,000\,000}{600\,000}$, which simplifies to $\frac{20\,000}{6}$.

Since the ground observatory radius is about $\frac{20\,000}{6}$ times the unaided eye radius, the ground observatory volume is about $\left(\frac{20\,000}{6}\right)^3 \doteq 3.7 \times 10^{10}$ times the unaided eye volume.

Linking Ideas: Mathematics and History
Archimedes of Syracuse, page 332

1. The volume of a sphere, radius r, is given by $V_s = \frac{4}{3}\pi r^3$.

The volume of a cylinder, radius r and height h, is given by $V_c = \pi r^2 h$.

When the sphere is inscribed in a cylinder, $h = 2r$. Substitute for h in the formula for V_c.

$$\begin{aligned}V_c &= \pi r^2(2r) \\ &= 2\pi r^3\end{aligned}$$

Selected Solutions — Chapter 6

Write the ratio $\frac{V_s}{V_c} = \frac{\frac{4}{3}\pi r^3}{2\pi r^3}$

$= \frac{4}{6}$

$= \frac{2}{3}$

Hence, the volume of the sphere is $\frac{2}{3}$ the volume of the cylinder.

The surface area of a sphere, radius r, is given by $A_s = 4\pi r^2$.
The surface area of a cylinder, radius r and height h, is given by $A_c = 2\pi rh + 2\pi r^2$.
Substitute $h = 2r$.

$$\begin{aligned} A_c &= 2\pi r(2r) + 2\pi r^2 \\ &= 4\pi r^2 + 2\pi r^2 \\ &= 6\pi r^2 \end{aligned}$$

Write the ratio.

$$\begin{aligned} \frac{A_s}{A_c} &= \frac{4\pi r^2}{6\pi r^2} \\ &= \frac{2}{3} \end{aligned}$$

Hence, the surface area of the sphere is $\frac{2}{3}$ the surface area of the cylinder.

6.2 Exercises, page 335

8. Answers may vary. For part f: I multiplied and divided $4m^4$ by m to get $\frac{4m^5}{m}$.

I multiplied and divided $4m^4$ by $3n$ to get $\frac{12m^4n}{3n}$.

I multiplied and divided $4m^4$ by $2ab$ to get $\frac{8abm^4}{2ab}$.

14. b) $\frac{\pi}{4}$; the ratio of the areas is equal to the ratio of the perimeters.

15. b) $\frac{6}{\pi}$; the ratio of the volumes is equal to the ratio of the surface areas.

Modelling Containers for Tennis Balls

Answers may vary. The balls in a real can do not fit snugly. Assume there is an extra 1 cm in the width, and in the length of the can. This means the radius is increased by 0.5 cm to 4.1 cm. The length is $(6 \times 3.6 + 1)$ cm, or 22.6 cm.

The volume of the can is $\pi(4.1)^2(22.6)$ cm^3, or approximately 1193.5 cm^3.

Then, the fraction of space occupied by the balls is $\frac{586.3}{1193.5}$, or approximately 0.5.

An increase of 1 cm in each dimension of the can reduces the fraction of space occupied by the balls from $\frac{2}{3}$ to about $\frac{1}{2}$.

Selected Solutions — Chapter 6

20. The volume of a hemisphere, radius r, is

$\frac{1}{2}(\frac{4}{3}\pi r^3) = \frac{2}{3}\pi r^3$.

The volume of a cone, radius r and height r, is

$\frac{1}{3}\pi r^2(r) = \frac{1}{3}\pi r^3$.

$$\frac{\text{Volume of hemisphere}}{\text{Volume of cone}} = \frac{\frac{2}{3}\pi r^3}{\frac{1}{3}\pi r^3} = \frac{2}{1}$$

The ratio volume of hemisphere : volume of cone is 2 : 1.

Mathematical Modelling: Could a Giant Survive?, page 338

2. a) The volume grows more rapidly than surface area as n increases. From the table, the volume increases from 1 cm^3 to 343 cm^3, while the surface area increases from 6 cm^2 to 294 cm^2.

b) As n increases, the ratio of volume to surface area increases. From the table, the ratio increases from $\frac{1}{6}$ to $\frac{7}{6}$.

c) From the last line in the table, $\frac{\text{Volume}}{\text{Surface area}} = \frac{n^3}{6n^2}$

$$= \frac{n}{6}$$

3. a) If the dimensions of an object are multiplied by n, the surface area of the object is multiplied by n^2, and the volume is multiplied by n^3. The giant would have a surface area $12^2 = 144$ times as great, and a volume $12^3 = 1728$ times as great.

b) The person's ratio is $\frac{V}{A}$. The giant's ratio would be $\frac{1728V}{144A} = \frac{12V}{A}$. The giant's ratio is 12 times the person's ratio.

4. a) The surface area would be 32^2, or 1024 times as great. The volume would be 32^3, or 32 768 times as great.

$32^2 = 1024$

$32^3 = 32\ 768$

b) The person's ratio is $\frac{V}{A}$. The giant's ratio is

$\frac{32\ 768V}{1024A} = \frac{32V}{A}$.

The giant's ratio is 32 times the person's ratio.

5. Answers may vary.

The $\frac{\text{Surface area}}{\text{Volume}}$ ratio decreases as a person gets larger. A giant would have trouble keeping cool. Giants might have trouble breathing, since they would have more cells requiring oxygen. The cross-sectional area of the bones does not increase as rapidly as the volume, so the bones may not support a giant.

Selected Solutions — Chapter 6

6. Answers may vary.

a) They probably lived very uncomfortable and short lives. Their skeletal system could not handle their relative sizes.

b) Nothing was made to fit these people. Tools, clothes, furniture — all these could contribute to accidents or illness.

7. Answers may vary.

a) Rectangular prism

Width (cm)	Length (cm)	Height (cm)	Surface area (cm^2)	Volume (cm^3)	$\frac{\text{Volume}}{\text{Surface area}}$
1	2	6	40	12	0.3
2	4	12	160	96	0.6
3	6	18	360	324	0.9
4	8	24	640	768	1.2
n	$2n$	$6n$	$40n^2$	$12n^3$	$0.3n$

Volume increases more rapidly than surface area as n increases. The ratio of volume to surface area increases as n increases. But volume is less than surface area until $n = 4$. If the person is very tall, he or she may have problems with his or her transpiration, respiration, and skeletal system. But people of average height should not have any problems.

Cylinder

Radius (cm)	Height (cm)	Surface area (cm^2)	Volume (cm^3)	$\frac{\text{Volume}}{\text{Surface area}}$
1	6	14π	6π	$\frac{3}{7}$
2	12	56π	48π	$\frac{6}{7}$
3	18	126π	162π	$\frac{9}{7}$
4	24	224π	384π	$\frac{12}{7}$
n	$6n$	$14\pi n^2$	$6\pi n^3$	$\frac{3n}{7}$

Volume increases more rapidly than surface area as n increases. The ratio of volume to surface area increases as n increases. But volume is less than surface area until $n = 2$. If a person is very tall, he or she may have problems with his or her transpiration, respiration, and skeletal system.

b) Numerical answers will differ, but conclusions probably will not.

6.3 Exercises, page 341

1. c) Yes. Let the numbers be a and b, where $a > b$.
Their sum is $a + b$. Their difference is $a - b$.
Add sum and difference:
$a + b + a - b = 2a$

Selected Solutions — Chapter 6

2. **b)** Yes. Let the numbers be a and b, where $a > b$.
   Their sum is $a + b$. Their difference is $a - b$.
   Subtract difference from sum.
   $a + b - (a - b) = 2b$

4. To find the other polynomial, I subtracted the given polynomial from the sum. That is, $15a + 4 - (3a - 6) = 15a + 4 - 3a + 6$, or $12a + 10$. This is the other polynomial.

7. To find the other polynomial, I subtracted the given polynomial from the sum. That is,
   $4x^2 - 7x + 3 - (-5x^2 - 8x + 5) = 4x^2 - 7x + 3 + 5x^2 + 8x - 5$, or $9x^2 + x - 2$. This is the other polynomial.

10. Simplifying each expression means fewer calculations when substituting, and less room for error.

14. To find the sum of the other two polynomials, I subtracted the given polynomial from the sum. That is,
    $-5x^2 - 8x + 5 - (4x^2 - 7x + 3) = -5x^2 - 8x + 5 - 4x^2 + 7x - 3$, or $-9x^2 - x + 2$.
    To find one polynomial, I chose any polynomial, such as $x^2 + x + 1$, and subtracted it from this sum.
    That is, $-9x^2 - x + 2 - (x^2 + x + 1) = -9x^2 - x + 2 - x^2 - x - 1$, or $-10x^2 - 2x + 1$.
    There is an infinite number of possibilities for the other two polynomials.
    I chose $-10x^2 - 2x + 1$ and $x^2 + x + 1$.

17. For part a: the polynomial has degree 4 and 3 terms.
    I first chose a term with degree 4. This means the sum of the exponents of the variables is 4. I chose x^2y^2. Then I chose two more terms with lesser degree: $+x$ and $+y$. The polynomial is $x^2y^2 + x + y$.

Investigate, page 344

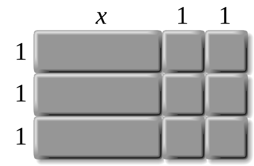
1. **c)** Both sides of the equation $2(x + 3) = 2x + 6$ represent the area of the rectangle. The left side is the product of the length $x + 3$ and the width 2. The right side is the sum of the areas of the rectangles, $x + x$, and the areas of 6 one-unit squares.

2. **c)** Both sides of the equation $x(x + 3) = x^2 + 3x$ represent the area of the rectangle. The left side is the product of the length $x + 3$ and the width x. The right side is the sum of the areas of the square, x^2, and 3 rectangles each x square units.

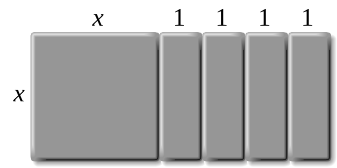
3. The results apply in all cases.

Selected Solutions — Chapter 6

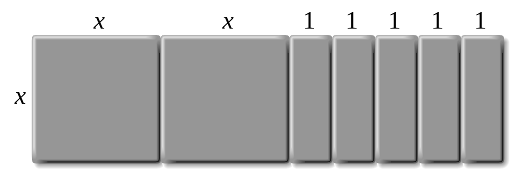
4. a) $3x + 6$



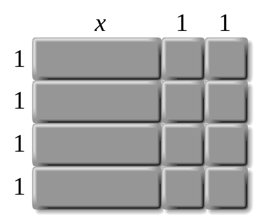
b) $x^2 + 4x$



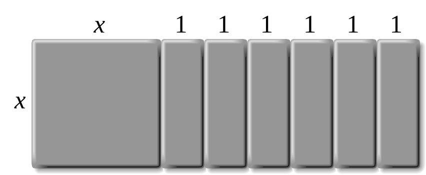
c) $2x^2 + 5x$



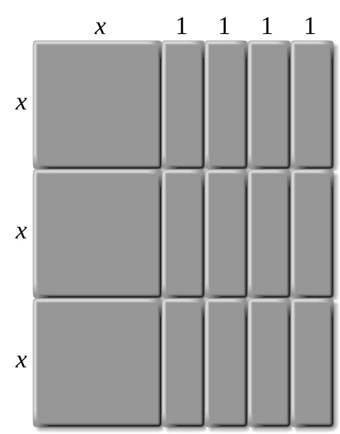
5. a) Answers may vary. $4(x + 2)$



b) $x(x + 6)$



c) Answers may vary. $3x(x + 4)$



Selected Solutions — Chapter 6

6.4 Exercises, page 347

3. Answers may vary. For part g: 3 is a common factor of $3x^2$ and $6x$. x is a common factor of $3x^2$ and $6x$. $3x$ is the greatest common factor of $3x^2$ and $6x$. I divided $3x^2$ by $3x$ to get x and I divided $6x$ by $3x$ to get 2.
$3x^2 + 6x = 3x(x + 2)$

8. a) Since the box is 10 cm wide and a square is cut from each corner, the size of the square has to be less than 5 cm. The size of the square has to be greater than 0 cm for the square to exist.
When x is small, for example, 0.5 cm, the box will have a relatively large base area and small height. As x increases, the base decreases in area and the height increases.
When x is large, for example, 4.5 cm, the box will have a small base area and a great height.

b) All boxes do not have the same surface area — as x increases the surface area decreases, because you are removing material from the box.

9. Answers may vary.

Side length of square cut out (cm)	Total surface area (cm^2)
1	196
2	184
3	164
4	136

Modelling the Surface Area of a Box

The model is a rectangle with length 20 cm and width 10 cm, with a square of length x centimetres cut from each corner. The real design might allow for tabs for gluing or inserting into slots. The modified design would use more cardboard. If the square was not removed, but simply cut along one side and tucked inside the box, then glued, more cardboard would be used, but the surface area wouldn't change.

15. Answers may vary. For part e: $10y(x - 3) + 7(x - 3)$ has a common factor $(x - 3)$. I wrote this factor, then divided each term by this factor, writing the quotient in brackets to get $(x - 3)(10y + 7)$.

20. Answers may vary. For part b: The height is 3 times the radius. Let the radius be r, then the height h is $3r$.
The general formula for the surface area of a cylinder, height h and radius r, is
$A = 2\pi r(h + r)$
Substitute $h = 3r$.

Selected Solutions — Chapter 6

$A = 2\pi r(3r + r)$
$= 2\pi r(4r)$
$= 8\pi r^2$

The surface area of the cylinder is $8\pi r^2$ units^2.

25. a) $2m(a - b) - 3n(b - a) - 7(a - b)$
Rewrite the middle term $-3n(b - a)$ as $+3n(a - b)$.
$2m(a - b) + 3n(a - b) - 7(a - b)$
The binomial $(a - b)$ is the common factor. Remove it.
$(a - b)(2m + 3n - 7)$

b) $2x^2(3a - 2b) + 5x(3a - 2b) - 9(2b - 3a)$
Write the last term $-9(2b - 3a)$ as $+9(3a - 2b)$.
$2x^2(3a - 2b) + 5x(3a - 2b) + 9(3a - 2b)$
The binomial $(3a - 2b)$ is the common factor. Remove it.
$(3a - 2b)(2x^2 + 5x + 9)$

c) $6a(b - a) + 4b(a - b) - 7(a - b)$
Write the first term $6a(b - a)$ as $-6a(a - b)$.
$-6a(a - b) + 4b(a - b) - 7(a - b)$
The binomial $(a - b)$ is the common factor. Remove it.
$(a - b)(-6a + 4b - 7)$

26. For each expression, factor the first two terms, factor the last two terms, then look for a common factor.

a) $x^2 + 3x + xy + 3y = x(x + 3) + y(x + 3)$
$= (x + 3)(x + y)$

b) $x^3 + x^2 + x + 1 = x^2(x + 1) + 1(x + 1)$
$= (x + 1)(x^2 + 1)$

c) $5am + a + 10bm + 2b = a(5m + 1) + 2b(5m + 1)$
$= (5m + 1)(a + 2b)$

d) $3x^2 - 6xy + 5x - 10y = 3x(x - 2y) + 5(x - 2y)$
$= (x - 2y)(3x + 5)$

e) $5m^2 + 10mn - 3m - 6n = 5m(m + 2n) - 3(m + 2n)$
$= (m + 2n)(5m - 3)$

f) $2a^2 - 6ab - 3a + 9b = 2a(a - 3b) - 3(a - 3b)$
$= (a - 3b)(2a - 3)$

Problem Solving: Round Robin Scheduling, page 352

1. a) Team A plays teams B, C, D, E. 4 games
This is true for every team. 20 games
But this counts every game twice.
Hence, there must be 10 games.

Selected Solutions — Chapter 6

The schedule has 25 spaces, but 5 of these cannot be used because a team cannot play itself. There are $25 - 5 = 20$ spaces for games. But this counts every game twice. Hence, there must be 10 games.

	A	B	C	D	E
A	■				
B		■			
C			■		
D				■	
E					■

b) Team A plays teams B, C, D, E, F. 5 games

This is true for every team. 30 games

But this counts every game twice.
Hence, there must be 15 games.

The schedule has 36 spaces, but 6 of these cannot be used because a team cannot play itself. There are $36 - 6 = 30$ spaces for games. But this counts every game twice. Hence, there must be 15 games.

	A	B	C	D	E	F
A	■					
B		■				
C			■			
D				■		
E					■	
F						■

2. a) Team 1 plays teams 2, 3, 4, …, n. $(n - 1)$ games

This is true for every team. $n(n - 1)$ games

But this counts every game twice.

Hence, there must be $\frac{n(n-1)}{2}$ games.

b) The schedule has $n \times n = n^2$ spaces, but n of these cannot be used because a team cannot play itself. There are $n^2 - n$ spaces for games. But this counts each game twice.

There must be $\frac{n^2 - n}{2}$ games.

c) Since $n^2 - n = n(n - 1)$, then $\frac{n^2 - n}{2} = \frac{n(n-1)}{2}$

The two formulas are equivalent.

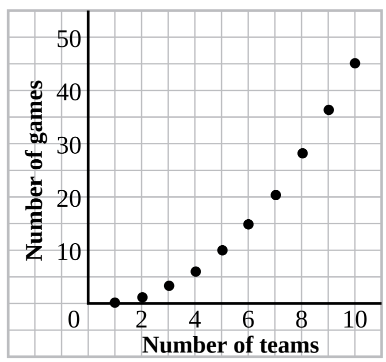
Selected Solutions — Chapter 6

3. a)

Number of teams, *n*	Number of games $\frac{n^2 - n}{2}$
1	0
2	1
3	3
4	6
5	10
6	15
7	21
8	28
9	36
10	45

b)

Number of games in a round robin tournament



4. a) This problem can be solved the same way as the Round Robin Tournament problem. Each vertex is connected to every other vertex, but not to itself, by a line. The first diagram has 9 vertices.

The first vertex connects to 8 other vertices: 8 line segments

This is the same for all vertices:
9×8 line segments $=$ 72 line segments

But this counts each line twice. Hence, there are 36 line segments.

The second diagram has 12 vertices.

The first vertex connects to 11 other vertices: 11 line segments

This is the same for all vertices:
12×11 line segments $=$ 132 line segments

But this counts each line twice. Hence, there are 66 line segments.

b) Using the formula from exercise 2, there would be $\frac{n(n-1)}{2}$ line segments.

Selected Solutions — Chapter 6

5. a)

Number of sides	Number of diagonals
3	0
4	2
5	5
6	9
7	14

b) i) $(n - 3)$ diagonals pass through each vertex of the polygon. There are n vertices. Hence, there are $n(n - 3)$ diagonals. But this counts each diagonal twice. Thus, there are $\frac{n(n-3)}{2}$ diagonals in a polygon with n sides.

ii) $\frac{n(n-1)}{2}$ line segments can be drawn to join the vertices of the polygon. n of these line segments are not diagonals; they are the sides of the polygon. Thus, the number of diagonals is

$$\begin{aligned}\frac{n(n-1)}{2} - n &= \frac{n^2 - n}{2} - \frac{2n}{2} \\ &= \frac{n^2 - 3n}{2} \\ &= \frac{n(n-3)}{2}\end{aligned}$$

6. Doubling the diameter of a circle quadruples the area. Thus, if a 20-cm pizza serves two people, then a 40-cm pizza serves 4×2 people $= 8$ people.

7. Since the polynomials are second-degree, each must contain at least one term with degree 2. The terms that have degree 2 are x^2, y^2, and xy. The polynomials may also contain terms of degree 1. There are x and y. List the possible polynomials systematically.

monomials: x^2; y^2; xy

binomials: $x^2 + y^2$; $x^2 + xy$; $x^2 + x$; $x^2 + y$;
$y^2 + xy$; $y^2 + x$; $y^2 + y$;
$xy + x$; $xy + y$

trinomials: $x^2 + y^2 + xy$; $x^2 + y^2 + x$; $x^2 + y^2 + y$;
$x^2 + xy + x$; $x^2 + xy + y$;
$x^2 + x + y$;
$y^2 + xy + x$; $y^2 + xy + y$;
$y^2 + x + y$;
$xy + x + y$

polynomials with 4 terms: $x^2 + y^2 + xy + x$; $x^2 + y^2 + xy + y$;
$x^2 + y^2 + x + y$; $x^2 + xy + x + y$;
$y^2 + xy + x + y$

polynomials with 5 terms: $x^2 + y^2 + xy + x + y$

There are 28 possible polynomials.

8. x is between 1 and 5.
y is between 5 and 10.

a) The least value of $x + y$ is $1 + 5$, or 6.
The greatest value of $x + y$ is $5 + 10$, or 15.
Hence, $6 \le x + y \le 15$

b) The least value of $x - y$ is $1 - 10$, or -9.
The greatest value of $x - y$ is $5 - 5$, or 0.
Hence, $-9 \le x - y \le 0$

c) The least value of xy is 1×5, or 5.
The greatest value of xy is 5×10, or 50.
Hence, $5 \le xy \le 50$

d) The least value of $\frac{x}{y}$ is $\frac{1}{10}$.
The greatest value of $\frac{x}{y}$ is $\frac{5}{5}$, or 1.
Hence, $\frac{1}{10} \le \frac{x}{y} \le 1$

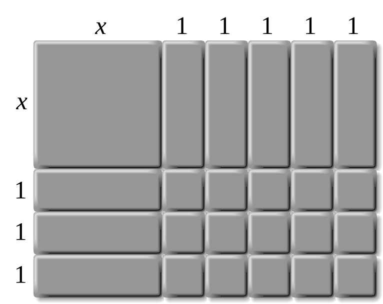
Investigate page 354

2. c) Both expressions, $(x + 4)(x + 1)$ and $x^2 + 5x + 4$, represent the area of the rectangle. The left side is the product of length $(x + 4)$ and width $(x + 1)$. The right side is the sum of the areas of the large square, x^2, the 5 rectangles, $5x$, and the 4 small unit squares, 4.
Hence, $(x + 4)(x + 1) = x^2 + 5x + 4$

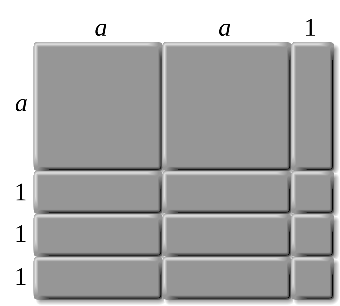
3. The result $(x + 4)(x + 1) = x^2 + 5x + 4$ applies in all cases.

4. In each case, the left side equals the right side.

5. a) $x^2 + 8x + 15$

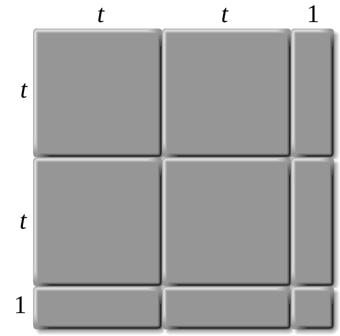


b) $2a^2 + 7a + 3$

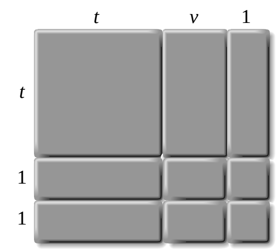


Selected Solutions — Chapter 6

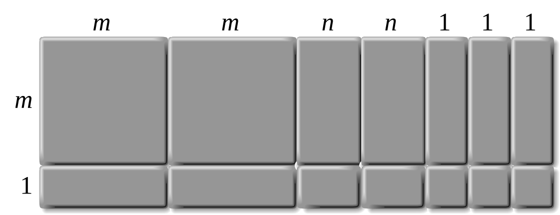
c) $4t^2 + 4t + 1$



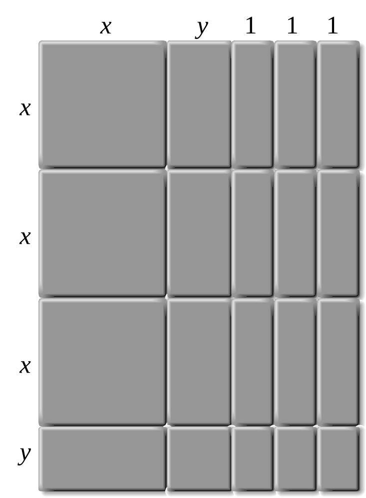
d) $t^2 + tv + 3t + 2v + 2$



e) $2m^2 + 2mn + 5m + 2n + 3$



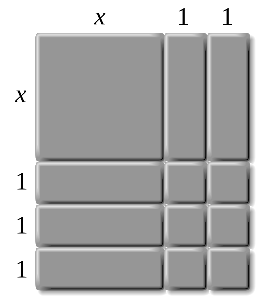
f) $3x^2 + 4xy + 9x + y^2 + 3y$



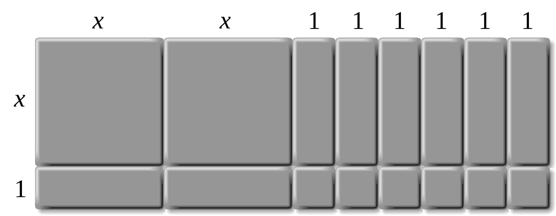
Selected Solutions — Chapter 6

6.5 Exercises, page 356

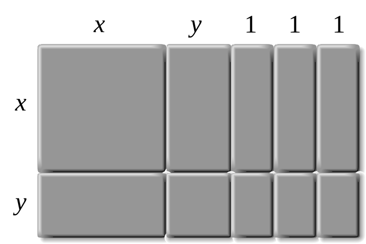
1. a) $(x + 3)(x + 2) = x^2 + 5x + 6$



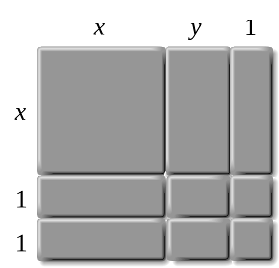
b) $(x + 1)(2x + 6) = 2x^2 + 8x + 6$



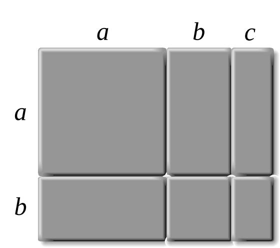
c) $(x + y)(x + y + 3) = x^2 + 2xy + y^2 + 3x + 3y$



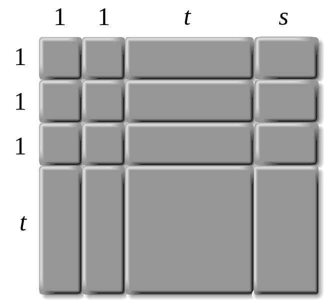
d) $(x + 2)(x + y + 1) = x^2 + xy + 3x + 2y + 2$



e) $(a + b)(a + b + c) = a^2 + 2ab + ac + b^2 + bc$



f) $(3 + t)(2 + t + s) = 6 + 5t + t^2 + 3s + st$



4. Answers may vary. For part a: To expand $(k + 2)^2$, I squared k to get k^2, I squared 2 to get 4 then I doubled $2 \times k$ to get $4k$. The expression is $k^2 + 4k + 4$.

5. Answers may vary.

a) If 10 is substituted for x in the algebraic multiplication, the result is the arithmetic multiplication. There are 4 multiplications in each. No place-holder 0 has to be inserted in the second multiplication.

b)

$$\begin{array}{r} 13 \\ \underline{\times\ 27} \\ 91 \\ \underline{260} \\ 351 \end{array} \qquad \begin{array}{r} x + \ \ 3 \\ \underline{\times\ 2x + \ \ 7} \\ 7x + 21 \\ \underline{2x^2 + 6x} \\ 2x^2 + 13x + 21 \end{array}$$

The steps are almost the same. No place-holder 0 has to be inserted in the second multiplication. In the second multiplication, like terms are not aligned. Terms are separated by plus signs.

6. Answers may vary.

a) The first term is x^2, the last term is -6 or 6.

b) The first term is x^2, the last term is -4 or 4.

c) The first term is x^2, the last term is -10 or 10.

d) The first term is x^2, the last term is -18 or 18.

8. a) i) The first term is always x^2.
The coefficients of the second terms form an arithmetic sequence with first term 2 and common difference 1.
The third terms form an arithmetic sequence with first term 1 and common difference 1.
Since 1 is added to the constant term in the second binomial, each time, this produces an increase of 1 in the coefficient of the middle term and in the third term.

ii) The first term is always x^2.
The coefficients of the second terms form an arithmetic sequence with first term -1 and common difference 1.
The third terms form an arithmetic sequence with first term -2 and common difference -2.

Since 1 is added each time to the constant term in the first binomial, and this is multiplied by –2 to get the third terms, they increase by –2.
Since 1 is added each time to the constant term in the first binomial, and this is added to –2 to get the coefficients of the second terms, they increase by 1.

iii) The first term is always x^2.
The coefficients of the second terms form an arithmetic sequence with first term 3 and common difference 1.
The third terms form an arithmetic sequence with first term 2 and common difference 2.
Since 1 is added each time to the constant term in the second binomial, and this is multiplied by 2 to get the constant terms, they increase by 2.
Since 1 is added each time to the constant term in the second binomial, and this is added to +2 to get the coefficients of the second terms, they increase by 1.

9. **a)** The first term is always t^2.
The second terms are the opposites of the square numbers.
In the binomials, since 1 is added to or subtracted from the constant terms, when these are multiplied, they form successive square numbers.

b) The first term is always t^2.
The second terms form a sequence similar to that in part a, except that they now all include the variable v^2, since v is a component of the second term in each binomial.

10. Answers may vary.

a) Calculate the area of the large rectangle $(x + y + x + 2y)(2x - y + x - y)$, then subtract the area of the small unshaded rectangle $(2x - y)(x + 2y)$.
Divide the shaded area into two smaller rectangles, by drawing a horizontal line, then add the areas of these rectangles:
$(x + y)(2x - y + x - y) + (x - y)(x + 2y)$

b) Calculate the area of the large rectangle $(2x + y + x + y)(x - 2y + 3x + y)$, then subtract the area of the small unshaded rectangle $(x - 2y)(2x + y)$.
Divide the shaded area into two smaller rectangles, by drawing a vertical line, then add the areas of these rectangles:
$(3x + y)(2x + y) + (x + y)(x - 2y + 3x + y)$

c) Divide the shaded area into two rectangles by drawing a vertical line, then add the areas of these rectangles:
$(5x + 4)(3x + 5 - x) + x(5x + 4 - 3x + 1)$
Complete a rectangle by extending the top side and right side of the figure. Calculate the area of the large rectangle $(5x + 4)(3x + 5)$, then subtract the area of the small unshaded rectangle $(3x - 1)(3x + 5)$.

Selected Solutions — Chapter 6

12. This is similar to exercise 8, page 348. All boxes will not have the same volume, because it is the product of length, width, and height.

13. Answers may vary.

Side length of square cut out (cm)	Volume (cm^3)
1	144
2	192
3	168
4	96

Modelling the Volume of a Box

This is similar to Modelling the Surface Area of a Box, on page 349. More cardboard would be required for flaps, but this would not affect the volume.

16. a) The coefficients form a pattern: 1, 2, 1; 1, 2, 2, 1; 1, 2, 2, 2, 1; … The pattern happens because, except for the first and last terms, each monomial containing x, or a power of x, occurs twice, and each time, there is one extra monomial that has the next highest power of x.

b) The first terms are x^2, x^3, x^4, … The second term alternates between −1 and +1. Each monomial product in x, except for the first and last terms, combines with its opposite to make zero.

22. a)

$$(x + 1)^2 = x^2 + 2x + 1$$

$$\begin{aligned}(x + 1)^3 &= (x + 1)(x + 1)^2 \\ &= (x + 1)(x^2 + 2x + 1) \\ &= x^3 + 2x^2 + x + x^2 + 2x + 1 \\ &= x^3 + 3x^2 + 3x + 1\end{aligned}$$

$$\begin{aligned}(x + 1)^4 &= (x + 1)(x + 1)^3 \\ &= (x + 1)(x^3 + 3x^2 + 3x + 1) \\ &= x^4 + 3x^3 + 3x^2 + x + x^3 + 3x^2 + 3x + 1 \\ &= x^4 + 4x^3 + 6x^2 + 4x + 1\end{aligned}$$

Put the coefficients in a triangle as follows:

```
        1   2   1
      1   3   3   1
    1   4   6   4   1
```

Each number is the sum of the two numbers above and to the right and left. Extend this pattern to determine the coefficients of the next 3 products.

Selected Solutions — Chapter 6

	1	5	10	10	5	1	
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

The products are:

$(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$
$(x+1)^6 = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$
$(x+1)^7 = x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$

b) $(x-1)^2 = x^2 - 2x + 1$
$(x-1)^3 = (x-1)(x-1)^2$
$= (x-1)(x^2 - 2x + 1)$
$= x^3 - 2x^2 + x - x^2 + 2x - 1$
$= x^3 - 3x^2 + 3x - 1$
$(x-1)^4 = (x-1)(x-1)^3$
$= (x-1)(x^3 - 3x^2 + 3x - 1)$
$= x^4 - 3x^3 + 3x^2 - x - x^3 + 3x^2 - 3x + 1$
$= x^4 - 4x^3 + 6x^2 - 4x + 1$

The pattern of coefficients is similar to that in part a, but the + and – signs alternate.

$(x-1)^5 = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$
$(x-1)^6 = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$
$(x-1)^7 = x^7 - 7x^6 + 21x^5 - 35x^4 + 35x^3 - 21x^2 + 7x - 1$

c) $(x+1)(x-1) = x^2 - 1$
$(x+1)^2(x-1)^2 = ((x+1)(x-1))^2$
$= (x^2 - 1)^2$
$= x^4 - 2x^2 + 1$
$(x+1)^3(x-1)^3 = ((x+1)(x-1))^3$
$= (x^2 - 1)^3$
$= (x^2 - 1)(x^2 - 1)^2$
$= (x^2 - 1)(x^4 - 2x^2 + 1)$
$= x^6 - 2x^4 + x^2 - x^4 + 2x^2 - 1$
$= x^6 - 3x^4 + 3x^2 - 1$

The pattern of coefficients is the same as in part b, but the exponents of the variables are double those in part b.

$(x+1)^4(x-1)^4 = x^8 - 4x^6 + 6x^4 - 4x^2 + 1$
$(x+1)^5(x-1)^5 = x^{10} - 5x^8 + 10x^6 - 10x^4 + 5x^2 - 1$
$(x+1)^6(x-1)^6 = x^{12} - 6x^{10} + 15x^8 - 20x^6 + 15x^4 - 6x^2 + 1$

23. a) The perimeter of the picture is 120 cm.
$2l + 2w = 120$
Divide each term by 2.
$l + w = 60$
Solve for l.
$l = 60 - w$

Selected Solutions — Chapter 6

b) $A = lw$

Substitute for l from part a.

$A = (60 - w)(w)$

$= 60w - w^2$

c) Let A_2 be the new area.

The new width is $w + 1$.

Replace w with $w + 1$ in the formula in part b.

$A_2 = 60(w + 1) - (w + 1)^2$

$= 60w + 60 - w^2 - 2w - 1$

$= 58w + 59 - w^2$

The change in area is

$A_2 - A = 58w + 59 - w^2 - (60w - w^2)$

$= 58w + 59 - w^2 - 60w + w^2$

$= -2w + 59$

24. Expressions may vary.

a) Convert the interest rate to a decimal: $r\% = \frac{r}{100}$.

The interest after 1 year is the amount invested multiplied by the interest rate.

The interest after 1 year is $100(\frac{r}{100}) = r$.

The amount after 1 year is the initial amount invested + interest: $100 + r$.

The interest after 2 years is $(100 + r)(\frac{r}{100})$

The amount after 2 years is $(100 + r) + (100 + r)(\frac{r}{100})$.

Remove $(100 + r)$ as a common factor.

$(100 + r)(1 + \frac{r}{100})$

This is usually written in a different form.

Remove 100 as a common factor from the first binomial.

$100(1 + \frac{r}{100})(1 + \frac{r}{100}) = 100(1 + \frac{r}{100})^2$

b) Expand $(100 + r)(1 + \frac{r}{100}) = 100 + r + r + \frac{r^2}{100}$

$= 100 + 2r + \frac{r^2}{100}$

100 is the initial investment.

$2r$ is twice the interest rate, as a percent.

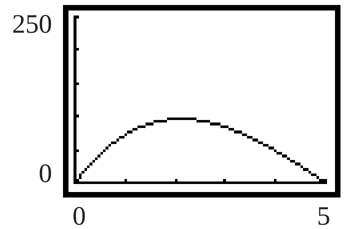
$\frac{r^2}{100}$ is the square of the interest rate, as a percent, divided by 100.

Together, $2r + \frac{r^2}{100}$ represent the interest on the investment.

Selected Solutions — Chapter 6

Exploring with a Graphing Calculator: Graphing the Surface Area and Volume of a Box, page 361

3. a) $V = 100x - 30x^2 + 2x^3$



b) The graphs have similar shapes, but the maximum volumes are different. The maximum volume of this graph is about half the maximum volume of the graph in exercise 2a.

6.6 Exercises, page 364

4. Answers may vary. For part k: To factor $s^2 - 12s + 20$, I used mental math to find two numbers whose product is 20 and whose sum is -12. I know both numbers are negative. I wrote $-1, -20$; then $-2, -10$, which is the pair I need.
$s^2 - 12s + 20 = (s - 2)(s - 10)$

9. Answers may vary. For part b: I could not factor $x^2 + 5x + 5$ because no two integers have both a sum and a product of 5. Five is a prime number, and its only factors are 1 and 5, which add to 6.

15. Answers may vary. For part d: To find the integral values of k for which $x^2 - 2x + k$ will factor, I know that k is the product of two numbers that add to make -2. The numbers could be:
$-1, -1$, then $k = 1$
$-3, 1$, then $k = -3$
$-4, 2$, then $k = -8$
$-5, 3$, then $k = -10$; and so on. There is an infinite number of possible values of k.

22. Answers may vary. For part f: To factor $3(2x + 4)^2 + 12(2x + 4)y - 36y^2$, I first replaced $2x + 4$ with the variable z to get $3z^2 + 12zy - 36y^2$. I removed 3 as a common factor, then wrote $3(z^2 + 4zy - 12y^2)$. I had to find two numbers that add to make 4 and multiply to make -12. Since the sum is $+4$, I know that the numerically larger number is positive. I tried $12, -1$; than $6, -2$, which is the required pair.
I wrote $3(z + 6y)(z - 2y)$, then replaced z with $2x + 4$ to get $3(2x + 4 + 6y)(2x + 4 - 2y)$. I noticed that each factor had 2 as a common factor, so I wrote
$3(2)(x + 2 + 3y)(2)(x + 2 - y)$, which becomes
$12(x + 2 + 3y)(x + 2 - y)$.

Selected Solutions — Chapter 6

24. Substitute $x = a + b$ and $y = a - b$ in each expression.

a) $x^2 + 2xy + y^2$

Factor first.

$x^2 + 2xy + y^2 = (x + y)^2$

Now substitute.

$$\begin{aligned}(x + y)^2 &= (a + b + a - b)^2 \\ &= (2a)^2 \\ &= 4a^2\end{aligned}$$

b) $x^2 - 5xy + 6y^2$

Factor first.

$x^2 - 5xy + 6y^2 = (x - 2y)(x - 3y)$

Now substitute.

$$\begin{aligned}&(x - 2y)(x - 3y) \\ &= (a + b - 2a + 2b)(a + b - 3a + 3b) \\ &= (-a + 3b)(-2a + 4b) \\ &= 2a^2 - 4ab - 6ab + 12b^2 \\ &= 2a^2 - 10ab + 12b^2\end{aligned}$$

c) $x^2 + 4xy - 12y^2$

Factor first.

$x^2 + 4xy - 12y^2 = (x + 6y)(x - 2y)$

Then substitute.

$$\begin{aligned}&(x + 6y)(x - 2y) \\ &= (a + b + 6a - 6b)(a + b - 2a + 2b) \\ &= (7a - 5b)(-a + 3b) \\ &= -7a^2 + 21ab + 5ab - 15b^2 \\ &= -7a^2 + 26ab - 15b^2\end{aligned}$$

d) $x^2y^2 - xy - 2$

Factor first.

$x^2y^2 - xy - 2 = (xy - 2)(xy + 1)$

Then substitute.

$$\begin{aligned}&(xy - 2)(xy + 1) \\ &= [(a + b)(a - b) - 2][(a + b)(a - b) + 1] \\ &= (a^2 - b^2 - 2)(a^2 - b^2 + 1) \\ &= a^4 - a^2b^2 + a^2 - a^2b^2 + b^4 - b^2 - 2a^2 + 2b^2 - 2 \\ &= a^4 - 2a^2b^2 - a^2 + b^2 + b^4 - 2\end{aligned}$$

25. a) $x^2 + 10x + 24 = (x + 4)(x + 6)$

$x^2 - 10x + 24 = (x - 4)(x - 6)$

$x^2 + 10x - 24 = (x - 2)(x + 12)$

$x^2 - 10x - 24 = (x + 2)(x - 12)$

b) Answers may vary. We need a number, such as 24, that has different factors, so that pairs of factors can add to make the same number.

Use guess and check. Consider 96. The factors are 1, 96; 2, 48; 3, 32; 4, 24; 6, 16; 8, 12.

The factors 4 and 24 have a difference of 20.

The factors 8 and 12 have a sum of 20.
Use these factors to write the trinomials.

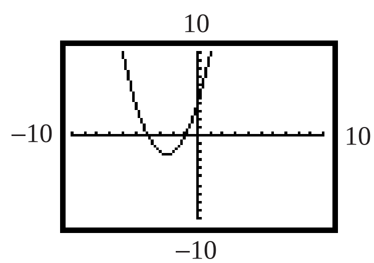
$(x + 8)(x + 12) = x^2 + 20x + 96$
$(x - 8)(x - 12) = x^2 - 20x + 96$
$(x + 24)(x - 4) = x^2 + 20x - 96$
$(x - 24)(x + 4) = x^2 - 20x - 96$

26. **a)** $x^2 + 5x + 6 = (x + 2)(x + 3)$
$x^2 + 6x + 5 = (x + 5)(x + 1)$

b) Answers may vary. We want two numbers such that the factors of one add to make the other number.
Use guess and check. Notice that 5 and 6 differ by 1. Try other pairs of numbers that differ by 1.
Try 10 and 11. $(x + 11)(x - 1) = x^2 + 10x - 11$
$(x - 10)(x - 1) = x^2 - 11x + 10$
Try 11 and 12. $(x + 12)(x - 1) = x^2 + 11x - 12$
$(x - 11)(x - 1) = x^2 - 12x + 11$
It appears that any pair of numbers that differ by 1 will work.

Linking Ideas: Verifying Trinomial Factorizations, page 368

1. $y = x^2 + 5x + 4$, $y = (x + 4)(x + 1)$



b) The two graphs are the same. This means that the two equations are equivalent.

Thus, $x^2 + 5x + 4 = (x + 4)(x + 1)$

c) The tables are the same.

x	*y*
0	4
1	10
2	18
3	28
4	40
5	54

x	*y*
0	4
1	10
2	18
3	28
4	40
5	54

d) Answers may vary.

Selected Solutions — Chapter 6

2. **b)** Cell B2: This is how to enter $x^2 - 7x + 10$ on a spreadsheet.

Cell C2: This is how to enter $(x - 2)(x - 5)$ on a spreadsheet.

Cell A3: This adds 1 to the previous cell, so that the values 1, 2, 3, 4, … are entered in the x column.

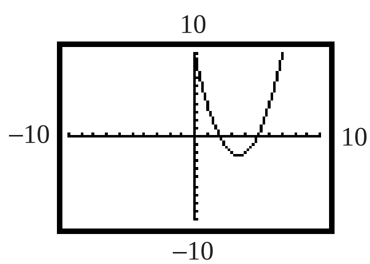
c)

	A	B	C
1	x	x^2 - 7x + 10	(x - 2)(x - 5)
2	1	4	6
3	2	0	0
4	3	-2	-2
5	4	-2	-2
6	5	0	0
7	6	4	4
8	7	10	10
9	8	18	18
10	9	28	28
11	10	40	40

d) The numbers in columns B and C are the same, no matter what starting value for x is used.

e) The same results are obtained for both equations. Thus, $x^2 - 7x + 10 = (x - 2)(x - 5)$

f) $x^2 - 7x + 10$, $(x - 2)(x - 5)$



g) Answers may vary.

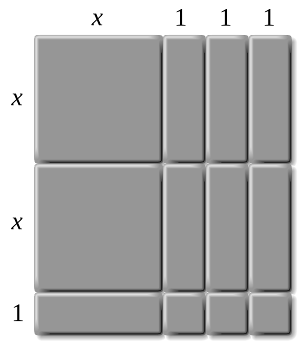
3. **a)** Yes, you could graph both the trinomial and the factorization to see if the graphs are the same.

b) Yes, you could graph the trinomial, then use the trace feature on the graphing calculator to determine where the graph intersects the x-axis. The opposite of each x-intercept is the number that occurs in each binomial when the trinomial is factored. Similarly, you could use the spreadsheet to graph the trinomial and identify the x-intercepts.

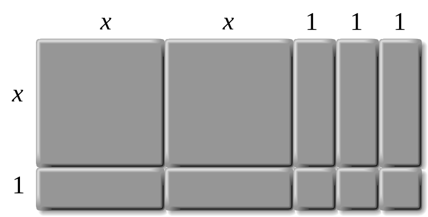
Selected Solutions — Chapter 6

Investigate: Algebra Tile Models of Trinomials, page 369

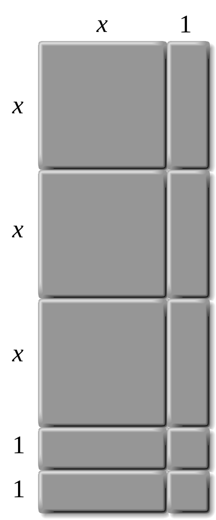
1. The length of the rectangle is $(2x + 1)$. The width of the rectangle is $(x + 3)$. Its area is $(2x + 1)(x + 3)$.
There are two x^2-tiles, seven x-tiles, and three 1-tiles. The area of the rectangle is $2x^2 + 7x + 3$.
Thus, $(2x + 1)(x + 3) = 2x^2 + 7x + 3$



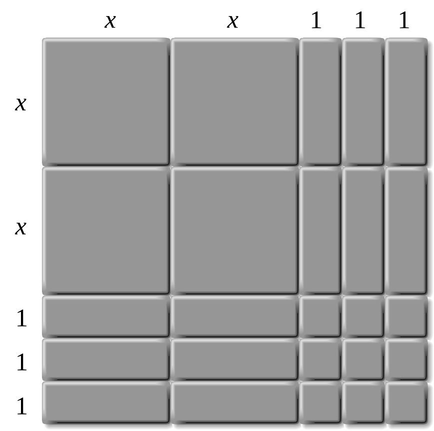
2. **a)** $2x^2 + 5x + 3 = (2x + 3)(x + 1)$



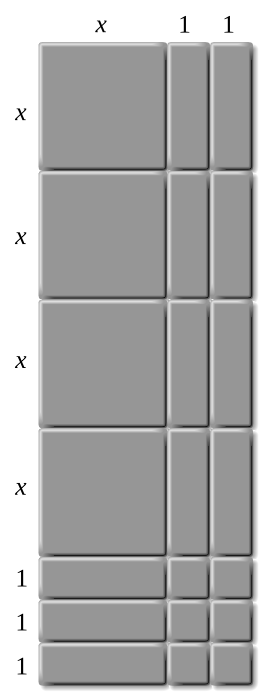
b) $3x^2 + 5x + 2 = (3x + 2)(x + 1)$



c) $4x^2 + 12x + 9 = (2x + 3)(2x + 3)$



d) $4x^2 + 11x + 6 = (4x + 3)(x + 2)$



3. Look at the trinomials in exercise 2. All have the form $ax^2 + bx + c$. When they are factored, there is a relationship among the numbers in the binomials.

For example, look at $2x^2 + 5x + 3$ that factors to $(2x + 3)(x + 1)$.

The x^2 term comes from the product of $2x$ and x — the first term of each binomial.

The x term comes from the sum of the products $2x \times 1$ and $3 \times x$, or $2x + 3x = 5x$.

The constant term comes from the product of 3 and 1, the second term of each binomial.

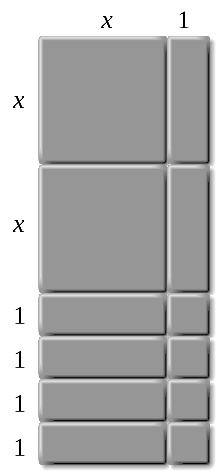
Selected Solutions — Chapter 6

4. $2x^2 + 6x + 4 = (2x + 4)(x + 1)$

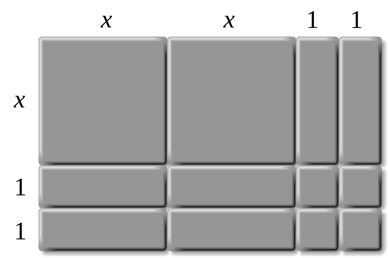
$= (x + 2)(2x + 2)$

You can factor this trinomial in two ways because it has a common factor.

The common factor 2 can be removed from the trinomial before further factoring, giving $2(x^2 + 3x + 2) = 2(x + 2)(x + 1)$. The 2 can be multiplied by either factor, giving the two factorizations above.



or



6.7 Exercises, page 371

6. In exercise 3, all the terms in the trinomials and the binomials are positive.
In exercise 4, in each trinomial, the second term is negative, and in each binomial the second term is negative.
In exercise 5, in each trinomial, the third term is negative, and in the binomials the second terms have different signs.

8. **a)** The factors are the same.

c) This happens with all squares of binomials.
Consider the general binomial square.

$$(ax + by)^2 = (ax + by)(ax + by)$$
$$= (ax)(ax) + (ax)(by) + (by)(ax) + (by)(by)$$
$$= a^2x^2 + 2abxy + b^2y^2$$

Consider how each term of the binomial is derived.

$$ax = \sqrt{a^2x^2}$$
$$by = \sqrt{b^2y^2}$$

14. **c)** Answers may vary. Begin with any two numbers, one of which is not prime.

Try 7 and 9. Use 7 as the coefficient of x^2 and 9 as the constant term in the trinomial.
The factors of 7 are 7 and 1 — these are the coefficients of x in the binomials.
The factors of 9 are +3, +3; −3, −3; +9, +1; and −9, −1.
These are the constant terms in the binomials. Write all possible

Selected Solutions — Chapter 6

combinations of the factors of 7 and the factors of 9.
$(7x + 3)(x + 3)$; $(7x - 3)(x - 3)$;
$(7x + 9)(x + 1)$; $(7x + 1)(x + 9)$;
$(7x - 9)(x - 1)$; $(7x - 1)(x - 9)$
Expand each trinomial.
$7x^2 + 24x + 9$; $7x^2 - 24x + 9$; $7x^2 + 16x + 9$;
$7x^2 + 64x + 9$; $7x^2 - 16x + 9$; $7x^2 - 64x + 9$

17. $(2x + 3)(3x + 2)$
$= (2x)(3x) + (2x)(2) + (3)(3x) + (3)(2)$
$= 6x^2 + 13x + 6$
To find whether a trinomial whose coefficients form a symmetrical pattern can be factored as two binomials whose coefficients are reversed, work backward. Let the binomials be $(ax + b)$ and $(bx + a)$.

$$(ax + b)(bx + a) = abx^2 + a^2x + b^2x + ab$$
$$= abx^2 + (a^2 + b^2)x + ab$$

The coefficients in the trinomial do form a symmetrical pattern.

18. a) $3xy^2 - 22xy + 6x$
There is a common factor x. Remove x.
$x(3y^2 - 22y + 6)$
No two integers have a sum of -22 and a product of 18. The trinomial cannot be factored further.

b) $3m^2n - 13mn + 12n$
There is a common factor n. Remove n.
$n(3m^2 - 13m + 12)$
Find two integers whose sum is -13 and whose product is 36. The integers are -4 and -9. Rewrite $-13m$ as $-4m - 9m$.
$n(3m^2 - 4m - 9m + 12)$
$= n(m(3m - 4) - 3(3m - 4))$
$= n(3m - 4)(m - 3)$

c) $4x^2y - 17xy - 15y$
There is a common factor y. Remove y.
$y(4x^2 - 17x - 15)$
Find two integers whose sum is -17 and whose product is -60. The integers are -20 and 3. Rewrite $-17x$ as $-20x + 3x$.
$y(4x^2 - 20x + 3x - 15)$
$= y(4x(x - 5) + 3(x - 5))$
$= y(x - 5)(4x + 3)$

d) $2x^3y + 7x^2y - 15xy$
There is a common factor xy. Remove xy.
$xy(2x^2 + 7x - 15)$
Find two integers whose sum is 7 and whose product is -30. The integers are 10 and -3. Rewrite $+7x$ as $+10x - 3x$.

$xy(2x^2 + 10x - 3x - 15)$
$= xy(2x(x + 5) - 3(x + 5))$
$= xy(x + 5)(2x - 3)$

e) $2m^3n - m^2n - 21mn$

There is a common factor mn. Remove mn.

$mn(2m^2 - m - 21)$

Find two integers whose sum is -1 and whose product is -42.
The integers are 6 and -7. Rewrite $-m$ as $+6m - 7m$.

$mn(2m^2 + 6m - 7m - 21)$
$= mn(2m(m + 3) - 7(m + 3))$
$= mn(m + 3)(2m - 7)$

f) $6x^3y - 7x^2y^2 - 3xy^3$

There is a common factor xy. Remove xy.

$xy(6x^2 - 7xy - 3y^2)$

Find two integers whose sum is -7 and whose product is -18.
The integers are -9 and 2. Rewrite $-7xy$ as $-9xy + 2xy$.

$xy(6x^2 - 9xy + 2xy - 3y^2)$
$= xy(3x(2x - 3y) + y(2x - 3y))$
$= xy(2x - 3y)(3x + y)$

19. a) $6x^3 + 33x^2 + 45x$

There is a common factor $3x$. Remove $3x$.

$3x(2x^2 + 11x + 15)$

Find two integers whose sum is 11 and whose product is 30.
The integers are 6 and 5. Rewrite $11x$ as $6x + 5x$.

$3x(2x^2 + 6x + 5x + 15)$
$= 3x(2x(x + 3) + 5(x + 3))$
$= 3x(x + 3)(2x + 5)$

b) $6a^3 + 26a^2 - 20a$

There is a common factor $2a$. Remove $2a$.

$2a(3a^2 + 13a - 10)$

Find two integers whose sum is 13 and whose product is -30.
The integers are 15 and -2. Rewrite $+13a$ as $+15a - 2a$.

$2a(3a^2 + 15a - 2a - 10)$
$= 2a(3a(a + 5) - 2(a + 5))$
$= 2a(a + 5)(3a - 2)$

c) $18x^2y - 3xy^2 - 45y$

There is a common factor $3y$. Remove $3y$.

$3y(6x^2 - xy - 15)$

The trinomial cannot be factored further.

d) $10m^3 - 25m^2 - 60m$

There is a common factor $5m$. Remove $5m$.

$5m(2m^2 - 5m - 12)$

Find two integers whose sum is -5 and whose product is -24.
The integers are -8 and 3. Rewrite $-5m$ as $-8m + 3m$.

$5m(2m^2 - 8m + 3m - 12)$
$= 5m(2m(m - 4) + 3(m - 4))$
$= 5m(m - 4)(2m + 3)$

e) $9a^3 - 39a^2 + 42a$
There is a common factor $3a$. Remove $3a$.
$3a(3a^2 - 13a + 14)$
Find two integers whose sum is -13 and whose product is 42.
The integers are -6 and -7. Rewrite $-13a$ as $-6a - 7a$.
$3a(3a^2 - 6a - 7a + 14)$
$= 3a(3a(a - 2) - 7(a - 2))$
$= 3a(a - 2)(3a - 7)$

f) $42ab^2 + 49ab - 28a$
There is a common factor $7a$. Remove $7a$.
$7a(6b^2 + 7b - 4)$
No two integers have a sum of 7 and a product of -24.
The trinomial cannot be factored further.

20. a) $32x^2 - 20x + 3$
Find two integers with a sum of -20 and a product of 96.
The integers are -12, -8. Rewrite $-20x$ as $-12x - 8x$.
$32x^2 - 12x - 8x + 3$
$= 4x(8x - 3) - 1(8x - 3)$
$= (8x - 3)(4x - 1)$

b) $24s^2 - 13s - 2$
Find two integers with a sum of -13 and a product of -48.
The integers are 3, -16. Rewrite $-13s$ as $-16s + 3s$.
$24s^2 - 16s + 3s - 2$
$= 8s(3s - 2) + 1(3s - 2)$
$= (3s - 2)(8s + 1)$

c) $4a^2 + 19a + 21$
Find two integers with a sum of 19 and a product of 84.
The integers are 7, 12. Rewrite $+19a$ as $+12a + 7a$.
$4a^2 + 12a + 7a + 21$
$= 4a(a + 3) + 7(a + 3)$
$= (a + 3)(4a + 7)$

d) $4x^2 - 21xy - 18y^2$
Find two integers with a sum of -21 and a product of -72.
The integers are -24, 3. Rewrite $-21xy$ as $-24xy + 3xy$.
$4x^2 - 24xy + 3xy - 18y^2$
$= 4x(x - 6y) + 3y(x - 6y)$
$= (x - 6y)(4x + 3y)$

e) $10a^2 - 19ab - 15b^2$
Find two integers with a sum of -19 and a product of -150.
The integers are 6, -25. Rewrite $-19ab$ as $-25ab + 6ab$.

Selected Solutions — Chapter 6

$10a^2 - 25ab + 6ab - 15b^2$
$= 5a(2a - 5b) + 3b(2a - 5b)$
$= (2a - 5b)(5a + 3b)$

f) $21x^2 + 25xy - 4y^2$
Find two integers with a sum of 25 and a product of -84.
The integers are 28, -3. Rewrite $+25xy$ as $+28xy - 3xy$.
$21x^2 + 28xy - 3xy - 4y^2$
$= 7x(3x + 4y) - y(3x + 4y)$
$= (3x + 4y)(7x - y)$

g) $21x^2 + 17x - 30$
Find two integers with a sum of 17 and a product of -30.
The integers are 35, -18. Rewrite $+17x$ as $+35x - 18x$.
$21x^2 + 35x - 18x - 30$
$= 7x(3x + 5) - 6(3x + 5)$
$= (3x + 5)(7x - 6)$

h) $72x^2 + 11x - 6$
Find two integers with a sum of 11 and a product of -432.
The integers are 27, -16. Rewrite $+11x$ as $+27x - 16x$.
$72x^2 + 27x - 16x - 6$
$= 9x(8x + 3) - 2(8x + 3)$
$= (8x + 3)(9x - 2)$

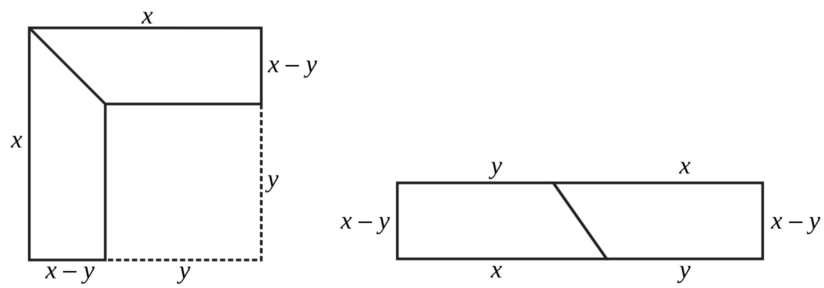
i) $15x^2 - 28x - 32$
Find two integers with a sum of -28 and a product of -480.
The integers are -40, 12. Rewrite $-28x$ as $+12x - 40x$.
$15x^2 + 12x - 40x - 32$
$= 3x(5x + 4) - 8(5x + 4)$
$= (5x + 4)(3x - 8)$

21. $ax^2 + bx + c$
From exercise 14, when a perfect square trinomial is factored, the two binomials are equal. The first term in each binomial is $\sqrt{a}$, and the second term is $\sqrt{c}$.
$ax^2 + bx + c = (\sqrt{a}x + \sqrt{c})(\sqrt{a}x + \sqrt{c})$
Expand the right side.
$= ax^2 + \sqrt{ac}x + \sqrt{ac}x + c$
$= ax^2 + 2\sqrt{ac}x + c$
Compare this with the general trinomial $ax^2 + bx + c$.
The coefficients of corresponding terms are equal.
Consider the x term. $b = 2\sqrt{ac}$
This is the relationship for a, b, and c.

Selected Solutions — Chapter 6

Investigate: Removing a Square from a Square, page 374

2. $x^2 - y^2$ and $(x - y)(x + y)$
Since the two figures have the same area,
$x^2 - y^2 = (x - y)(x + y)$



3. c) Consider the general expression.

$(ax - by)(ax + by)$
$= (ax)(ax) + (ax)(by) - (by)(ax) - (by)(by)$
$= a^2x^2 - b^2y^2$

A difference of squares always results when you multiply two binomials that are the sum and the difference of the same two quantities.

6.8 Exercises, page 376

2. a) Answers may vary.
Choose two rational numbers, 5.8 and 7.8.
Their product is $5.8 \times 7.8 = 45.24$
Their mean is $\frac{5.8 + 7.8}{2} = 6.8$
The square of the mean is $6.8^2 = 46.24$
Choose a negative and a positive integer, −1 and 1.
Their product is $-1 \times 1 = -1$
Their mean is $\frac{-1 + 1}{2} = 0$
The square of the mean is $0^2 = 0$
Choose two negative integers, −3 and −1.
Their product is $-3 \times (-1) = 3$
Their mean is $\frac{-3 + (-1)}{2} = -2$
The square of the mean is $(-2)^2 = 4$
Choose two irrational numbers, $\sqrt{2}$ and $\sqrt{2} + 2$.
Their product is $\sqrt{2}(\sqrt{2} + 2) = 2 + 2\sqrt{2}$
Their mean is $\frac{\sqrt{2} + \sqrt{2} + 2}{2} = \frac{2\sqrt{2} + 2}{2}$
$= \sqrt{2} + 1$
The square of the mean is $(\sqrt{2} + 1)^2 = 2 + 2\sqrt{2} + 1$
$= 3 + 2\sqrt{2}$
In each case, the square of the mean is 1 more than the product.

b) The conclusion is true for all real numbers. Let the numbers be x and $x + 2$.

Selected Solutions — Chapter 6

Their product is $x(x+2) = x^2 + 2x$

Their mean is $\frac{x+x+2}{2} = x + 1$

The square of the mean is $(x+1)^2 = x^2 + 2x + 1$

The square of the mean is 1 more than the product.

This result proves the conclusion.

6. Answers may vary. For part m: $(x-y)^2 - z^2$; I used the difference of squares where $a^2 - b^2 = (a-b)(a+b)$. In this case, $a = x - y$ and $b = z$. The expression becomes $(x-y-z)(x-y+z)$.

9. Answers may vary. Use guess and check.

$1x^2 + 0x - 1$ is

$x^2 - 1 = (x-1)(x+1)$

$-1x^2 + 0x + 1$ is

$1 - x^2 = (1-x)(1+x)$

$2x^2 + 1x + 0$ is

$2x^2 + x = x(2x+1)$

$-2x^2 - 1x - 0$ is

$-2x^2 - 1x = -x(2x+1)$

Modelling the Great Pyramid

Suppose the slant height and the height are both 20 cm, or 0.2 m greater than the dimensions given in exercise 11.

Then, $h = 146.8$ m and $s = 186.6$ m

$$\begin{aligned} V &= \tfrac{4}{3}h(s^2 - h^2) \\ &= \tfrac{4}{3}(146.8)(186.6^2 - 146.8^2) \\ &\doteq 2\,597\,248.2 \end{aligned}$$

From exercise 11b, volume is $2\,590\,598\ \text{m}^3$.

Subtract the two volumes: $2\,597\,248 - 2\,590\,598 = 6650$

The volumes differ by $6650\ \text{m}^3$.

The ratio of the height of the small piece to the height of the original is equal to the ratio of the slant height of the small piece to the slant height of the original. Use the new slant height and the height. Let the slant height of the missing piece be *s* metres.

$$\begin{aligned} \frac{9.1}{146.8} &= \frac{s}{186.6} \\ s &= \frac{9.1 \times 186.6}{146.8} \\ s &\doteq 11.6 \end{aligned}$$

Assume the missing piece is a square-based pyramid. Use the formula to determine its volume.

$$\begin{aligned} V &= \tfrac{4}{3}h(s^2 - h^2) \\ &= \tfrac{4}{3}(9.1)(11.6^2 - 9.1^2) \\ &= 627.9 \end{aligned}$$

The missing portion has a volume of approximately $630\ \text{m}^3$.

Assume that with the limestone surface the height and slant height were 50 cm, or 0.5 m greater.

Selected Solutions — Chapter 6

The height was $146.8 \text{ m} + 0.5 \text{ m} = 147.3 \text{ m}$
The slant height was $186.6 \text{ m} + 0.5 \text{ m} = 187.1 \text{ m}$
The volume was $V = \frac{4}{3}(147.3)(187.1^2 - 147.3^2)$
$\doteq 2\,613\,911$
Subtract the volume above from this volume to determine the volume of limestone removed.
$2\,613\,911 - 2\,597\,248 = 16\,663$
About $16\,700 \text{ m}^3$ of limestone were removed.
In exercise 11, the calculated volume was $2\,590\,598 \text{ m}^3$.
The actual volume is $(2\,597\,248 - 630) \text{ m}^3$, or $2\,596\,618 \text{ m}^3$.
The difference between this volume and the volume calculated in exercise 11 is $(2\,596\,618 - 2\,590\,598) \text{ m}^3$, or 6020 m^3.
As a percent of the volume in exercise 11, this is
$\frac{6020}{2\,590\,598} \times 100\% \doteq 0.23\%$
As a percent of the actual volume, this is
$\frac{6020}{2\,596\,618} \times 100\% \doteq 0.23\%$

12. **a)** The larger rectangle has area $x(x - y)$. The smaller rectangle has area $(x - y)y$. The total area is

$$\begin{aligned}(x - y)(y) + (x)(x - y) &= (x - y)(y + x) \\ &= x^2 - y^2\end{aligned}$$

b) The square has area $(x - y)(x - y)$. Each of the two rectangles has area $(x - y)y$. The total area is

$$\begin{aligned}&(x - y)(y) + (x - y)(y) + (x - y)(x - y) \\ &= (x - y)(y + y + x - y) \\ &= (x - y)(x + y) \\ &= x^2 - y^2\end{aligned}$$

13. For each trapezoid: the height is $\frac{x - y}{2}$; the parallel sides have lengths x and y.
The area of a trapezoid is given by $A = \frac{a + b}{2} \times h$, where a and b are the lengths of the parallel sides, and h is the height.
Substitute for a, b, and h.
The area of each trapezoid is $A = \left(\frac{x + y}{2}\right)\left(\frac{x - y}{2}\right)$
The shaded area is 4 times this.

$$\begin{aligned}4\left(\frac{x + y}{2}\right)\left(\frac{x - y}{2}\right) &= (x + y)(x - y) \\ &= x^2 - y^2\end{aligned}$$

14. **a)** $$\begin{aligned}8d^2 - 32e^2 &= 8(d^2 - 4e^2) \\ &= 8(d - 2e)(d + 2e)\end{aligned}$$

b) $25m^2 - \frac{1}{4}n^2 = (5m - \frac{1}{2}n)(5m + \frac{1}{2}n)$

c) $$\begin{aligned}18x^2y^2 - 50y^4 &= 2y^2(9x^2 - 25y^2) \\ &= 2y^2(3x - 5y)(3x + 5y)\end{aligned}$$

Selected Solutions — Chapter 6

d) e) Cannot be factored

f) $p^2 - \frac{1}{9}q^2 = (p - \frac{1}{3}q)(p + \frac{1}{3}q)$

g) $5x^4 - 80 = 5(x^4 - 16) = 5(x^2 + 4)(x^2 - 4)$
$= 5(x^2 + 4)(x + 2)(x - 2)$

h) $\frac{x^2}{16} - \frac{y^2}{49} = (\frac{x}{4} + \frac{y}{7})(\frac{x}{4} - \frac{y}{7})$

15. Answers may vary. For part d: $10a^2 - 7b^2$ is not a difference of squares, because 10 and 7 are not perfect squares. There are no common factors.

16. a) Use guess and check.
The first 10 perfect squares are 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100.
Equate each one to $5p + 1$.

$5p + 1 = 1$
$5p = 0$
$p = 0$ — This is not a prime number.

$5p + 1 = 4$
$5p = -3$
$p = \frac{-3}{5}$ — This is not a prime number.

$5p + 1 = 9$
$5p = 8$
$p = \frac{8}{5}$ — This is not a prime number.

$5p + 1 = 16$
$5p = 15$
$p = 3$ — This is a prime number.

$5p + 1 = 25$
$5p = 24$
$p = \frac{24}{5}$ — This is not a prime number.

$5p + 1 = 36$
$5p = 35$
$p = 7$ — This is a prime number.

$5p + 1 = 49$
$5p = 48$
$p = \frac{48}{5}$ — This is not a prime number.

We would have to check every perfect square to determine if p was a prime number.

b) From part a, there are 2 prime numbers, but there may be more.

c) Since $5p + 1$ is a perfect square, let $5p + 1 = n^2$, where n is a natural number.
Then, $5p + 1 = n^2$
$5p = n^2 - 1$
$5p = (n - 1)(n + 1)$
Compare the left side and the right side. Since p is prime, there are only these possibilities:

• $5 = n - 1$ and $p = n + 1$, or $5p = n - 1$ and $n + 1 = 1$
• $5 = n + 1$ and $p = n - 1$, or $5p = n + 1$ and $n - 1 = 1$

Take each one in turn.

$5 = n - 1$
$n = 6$

Therefore, $p = n + 1$
$= 6 + 1$
$= 7$

If $n + 1 = 1$, then $n = 0$, which is not possible

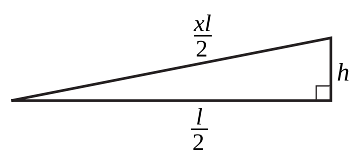
And, $5 = n + 1$
$n = 4$

Therefore, $p = n - 1$
$= 4 - 1$
$= 3$

If $n - 1 = 1$, $n = 2$, which is not possible

Hence, the only possible values of p are 3 and 7.

17. a) Consider half the strip. The diagram approximates a right triangle.



$$\left(\frac{xl}{2}\right)^2 = \left(\frac{l}{2}\right)^2 + h^2$$

$$h^2 = \left(\frac{xl}{2}\right)^2 - \left(\frac{l}{2}\right)^2$$

$$= \frac{x^2l^2}{4} - \frac{l^2}{4}$$

$$= \frac{l^2(x^2 - 1)}{4}$$

Therefore, h is approximately $\frac{\sqrt{l^2(x^2 - 1)}}{2}$.

b) Substitute $l = 250$ cm.

i) For steel, substitute $x = 1.0012$.

$$h = \frac{\sqrt{250^2(1.0012^2 - 1)}}{2}$$

$$= \frac{\sqrt{150.09}}{2}$$

$$\doteq 6.13$$

h is approximately 6.13 cm.

ii) For brass, substitute $x = 1.0020$.

$$h = \frac{\sqrt{250^2(1.0020^2 - 1)}}{2}$$

$$= \frac{\sqrt{250.25}}{2}$$

$$\doteq 7.91$$

h is approximately 7.91 cm.

Selected Solutions — Chapter 6

iii) For aluminum, substitute $x = 1.0024$.

$$h = \frac{\sqrt{(250^2(1.0024^2 - 1)}}{2}$$
$$= \frac{\sqrt{300.36}}{2}$$
$$\doteq 8.67$$

h is approximately 8.67 cm.

c) From the diagram, the curved strip is longer than twice the length of the hypotenuse, hence the corresponding value of h will be less than the estimate.

Modelling the Height of a Projectile, page 382

$h = ut - 0.5gt^2$

Substitute $u = 19.2$ and $g = 9.8$.

$h = 19.2t - 0.5(9.8)t^2$
$= 19.2t - 4.9t^2$

The height of the football, h metres, t seconds after it has been kicked with vertical speed 19.2 m/s, is given by
$h = 19.2t - 4.9t^2$.
Substitute $h = 15$.

$15 = 19.2t - 4.9t^2$

$4.9t^2 - 19.2t + 15 = 0$

Use guess and check to solve the equation.

Substitute values for t in the equation.

t	left side
0	15
1	0.7
2	–3.8
3	1.5

Look for values of t that make the left side zero.

There appears to be a value for t between 1 and 2 but closer to 1, and between 2 and 3 but closer to 3. Try some values of t close to 1 and close to 3.

t	left side
1.1	–0.191
1.2	–0.984
2.9	0.529
2.8	–0.344

The left side of the equation is approximately 0 for $t = 1.1$ s and $t = 2.8$ s.

The model does not consider air resistance. This may affect the height of the ball.

Selected Solutions — Chapter 6

6.9 Exercises, page 384

8. The box and can have equal volumes and equal heights. So, the base areas are equal.
For a square to have the same area as a circle, the side of the square must be less than the diameter of the circle. Suppose the square has side length l, and the circle has diameter d.
The area of the square is l^2.
The area of the circle is $\pi(\frac{d}{2})^2 = \frac{\pi d^2}{4} \doteq 0.78$

$$l^2 = \frac{\pi d^2}{4}$$
$$l = d\sqrt{\frac{\pi}{4}}$$
$$l \doteq 0.88d$$

6.10 Exercises, page 389

5. Answers may vary. For part d: I divided $(n + 5)$ into $n^2 - 11n + 6$.

$$\begin{array}{r|l} n - 16 & \\ n + 5 \,\big)\, n^2 - 11n + 6 & n \text{ into } n^2 \text{ is } n. \\ \underline{n^2 + 5n} \qquad & \text{Multiply } n + 5 \text{ by } n. \\ -16n + 6 & \text{Subtract and bring down 6.} \\ \underline{-16n - 80} & n \text{ into } -16n \text{ is } -16. \\ 86 & \text{Multiply } n + 5 \text{ by } -16\text{, then subtract.} \end{array}$$

The divisor is $n + 5$, the quotient is $n - 16$, the dividend is $n^2 - 11n + 6$, and the remainder is 86.
The division statement is $n^2 - 11n + 6 = (n + 5)(n - 16) + 86$.

10. Answers may vary.

a) For parts i and ii, there is no remainder. This means that the divisor is a factor of dividend, and the quotient is a factor of the dividend. That is, the quotient for part i is the divisor for part ii, and vice versa.
For parts iii and iv, the remainders are equal. This means the divisor for part iii is the quotient for part iv, and vice versa.

b) For the pattern in the quotient column: When the two binomials are multiplied, the sum of the two middle terms must be $7x$.
So, if the constant in one binomial is increased by 1, the constant in the other binomial must be decreased by 1.
For the pattern in the remainder column: When the two binomials are multiplied, the constant term is the product of the constant terms in the two binomials. There is a remainder in the division problem when these two products are less than 12.
Starting at the first line in the table, the products of the constant terms are: $0 \times 7 = 0$, $1 \times 6 = 6$, $2 \times 5 = 10$, $3 \times 4 = 12$, and so on. The products are 0, 6, 10, 12, and so on.
The remainders are the amounts left over when each product is subtracted from 12: 12, 6, 2, 0, and so on.

Selected Solutions — Chapter 6

11. c) These are the similarities and differences:
The quotients in the second column decrease by 1, just as in exercise 10.
The remainders in the third column decrease, then increase according to a pattern.
The patterns formed by the remainders are the same in exercise 10 and exercise 11a.
The patterns in exercises 11b and c are the same, but different from the other two.

13. Answers may vary.

a) For exercise 12a: In each case, the product of divisor and quotient is 3 less than the dividend. For example, $(x + 3)(2x + 3) = 2x^2 + 9x + 9$, which is 3 less than $2x^2 + 9x + 12$.
The coefficients of x form an arithmetic sequence 7, 8, 9, … , where the nth term is $n + 6$. The constant terms form an arithmetic sequence 6, 9, 12, … , where the nth term is $3n + 3$. Hence, the dividend in the nth row is $2x^2 + (n + 6)n + (3n + 3)$. In this row, the divisor is $x + 3$, and the quotient is $2x + n$. The product $(x + 3)(2x + n)$ equals $2x^2 + (n + 6)x + 3n$, which is 3 less than the expression for the dividend.
For exercise 12b: This is similar to exercise 12a. For example, the product of divisor and quotient is $(x + 2)(x^2 + 4x + 3) = x^3 + 6x^2 + 11x + 6$, which is 3 less than $x^3 + 6x^2 + 11x + 9$. As for exercise 12a, this can be proved in general using arithmetic sequences. Starting at the same place, the nth term of the sequences formed by the coefficients of x and the constant terms are $n + 8$ and $2n + 3$. Hence, the dividend in the nth row is $x^3 + 6x^2 + (n + 8)x + (2n + 3)$. In this row, the divisor is $x + 2$, and the quotient is $x^2 + 4x + n$. The product $(x + 2)(x^2 + 4x + n)$ equals $x^3 + 6x^2 + (n + 8)x + 2n$, which is 3 less than the expression for the dividend.

14. Answers may vary.

a) In each case, the product of divisor and quotient is the first two terms of the dividend and a negative third term that requires a perfect square to be added to it to give the constant term 1 in the dividend. For example, the product $(x + 3)(x - 1)$ equals $x^2 + 2x - 3$, which is 4 less than $x^2 + 2x + 1$. This can also be proved in general. In the nth row, the divisor is $x + n$, and the quotient is $x + (2 - n)$. The product $(x + n)(x + 2 - n)$ equals $x^2 + 2x + 2n - n^2$, which is $n^2 - 2n + 1$, or $(n - 1)^2$ less than $x^2 + 2x + 1$.

b) In the nth row, the divisor is $nx + 1$ and the coefficient of x in the quotient is $\frac{1}{n}$. The constant term in the quotient is a fraction with numerator $2n - 1$ and denominator n^2. Hence, the quotient in the

nth row is $\frac{1}{n}x + \frac{2n-1}{n^2}$. The product of the divisor and the quotient is $(nx+1)(\frac{1}{n}x + \frac{2n-1}{n^2})$, which equals $x^2 + 2x + \frac{2n-1}{n^2}$. The first two terms are the same as those in the dividend. The difference between 1 and the third term is $1 - \frac{2n-1}{n^2}$, which simplifies to $\frac{(n-1)^2}{n^2}$, which is the expression for the remainder in the nth row.

16. Answers may vary. For part a: I divided $x^2 + 5x + 6$ by $2x + 3$.

$$\begin{array}{r@{\,}l}
 & 0.5x + 1.75 \\
2x + 3 & \overline{)\, x^2 + 5x \quad + 6} \\
 & \underline{x^2 + 1.5x} \\
 & 3.5x + 6 \\
 & \underline{3.5x + 5.25} \\
 & 0.75
\end{array}$$

$2x$ into x^2 is $0.5x$. Multiply $2x + 3$ by $0.5x$, then subtract. Bring down 6. Divide $2x$ into $3.5x$ to get 1.75. Multiply $2x + 3$ by 1.75, then subtract.

The quotient is $0.5x + 1.75$ with a remainder of 0.75. Since there is a remainder, the divisor is not a factor.

18. a) $\frac{x^2-1}{x-1} = x + 1$, since $x^2 - 1$ can be factored as $(x-1)(x+1)$

$$\begin{array}{r@{\,}l}
 & x^2 + x + 1 \\
x - 1 & \overline{)\, x^3 + 0x^2 + 0x - 1} \\
 & \underline{x^3 - x^2} \\
 & x^2 + 0x \\
 & \underline{x^2 - x} \\
 & x - 1
\end{array}$$

$\frac{x^3-1}{x-1} = x^2 + x + 1$

$$\begin{array}{r@{\,}l}
 & x^3 + x^2 + x + 1 \\
x - 1 & \overline{)\, x^4 + 0x^3 + 0x^2 + 0x - 1} \\
 & \underline{x^4 - x^3} \\
 & x^3 + 0x^2 \\
 & \underline{x^3 - x^2} \\
 & x^2 + 0x \\
 & \underline{x^2 - x} \\
 & x - 1
\end{array}$$

$\frac{x^4-1}{x-1} = x^3 + x^2 + x + 1$

For each expression, $x - 1$ is a factor, and the quotient contains all non-negative powers of x less than the power of x in the dividend. These occur because at each step we subtract a negative power of x from 0, which makes the power positive, then we divide by x. Following this pattern, the next 3 expressions are

Selected Solutions — Chapter 6

$\frac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1$

$\frac{x^6 - 1}{x - 1} = x^5 + x^4 + x^3 + x^2 + x + 1$

$\frac{x^7 - 1}{x - 1} = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$

The explanation is that the products of divisors and quotients all multiply to produce the dividends, with nothing left over.

For example, the product

$(x - 1)(x^3 + x^2 + x + 1)$ consists of a first term of x^4 and a last term of -1, and all the other terms occur in pairs with opposite signs such as $-x^3$ and $+x^3$. Hence,

$(x - 1)(x^3 + x^2 + x + 1) = x^4 - 1$

b)

$$\begin{array}{r|l} & x \\ \hline x+1 & x^2 + x + 1 \\ & \underline{x^2 + x} \\ & \qquad 0 + 1 \end{array}$$

$\frac{x^2 + x + 1}{x + 1} = x,\ \text{R1}$

$$\begin{array}{r|l} & x^2 + \qquad 1 \\ \hline x+1 & x^3 + x^2 + x + 1 \\ & \underline{x^3 + x^2} \\ & \qquad 0 + x + 1 \end{array}$$

$\frac{x^3 + x^2 + x + 1}{x + 1} = x^2 + 1$

$$\begin{array}{r|l} & x^3 + \qquad x \\ \hline x+1 & x^4 + x^3 + x^2 + x + 1 \\ & \underline{x^4 + x^3} \\ & \qquad 0 + x^2 + x \\ & \qquad \underline{x^2 + x} \\ & \qquad\qquad 0 + 1 \end{array}$$

$\frac{x^4 + x^3 + x^2 + x + 1}{x + 1} = x^3 + x,\ \text{R1}$

The quotients are alternate terms of the dividend, beginning with the second term each time.

When the first term of the dividend is an even power of x, the remainder is 1.

When the first term of the dividend is an odd power of x, there is no remainder; the divisor is a factor of the dividend.

Following this pattern, the next 3 expressions are:

$\frac{x^5 + x^4 + x^3 + x^2 + x + 1}{x + 1} = x^4 + x^2 + 1$

$\frac{x^6 + x^5 + x^4 + x^3 + x^2 + x + 1}{x + 1} = x^5 + x^3 + x,\ \text{R1}$

$\frac{x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1}{x + 1} = x^6 + x^4 + x^2 + 1$

Again, consider the products of the divisors and the quotients.

This time there are two cases, depending on whether the highest power in the dividend is odd or even.

If the highest power in the dividend is odd, the highest power in the quotient is even. The products of divisors and quotients

multiply to produce the dividend. For example, in $(x + 1)(x^4 + x^2 + 1)$, you get $x^5 + x^3 + x$ by multiplying the first term of the first factor by the second factor, and you get $x^4 + x^2 + 1$ by multiplying the second term of the first factor by the second factor. Adding the results gives the dividend.
If the highest power in the dividend is even, the highest power in the quotient is odd. In this case, the products of divisors and quotients multiply to produce 1 less than the dividend. For example, in $(x + 1)(x^5 + x^3 + x)$, you get $x^6 + x^4 + x^2$ by multiplying the first term of the first factor by the second factor, and you get $x^5 + x^3 + x$ by multiplying the second term of the first factor by the second factor. Adding the results gives $x^6 + x^5 + x^4 + x^3 + x^2 + x$, which is 1 less than the dividend. Hence, the remainder is 1.

19. The pattern is $\frac{x^2 - 1}{x + 1} = x - 1$

$\frac{x^3 - 1}{x + 1} = x^2 - x + 1, \text{R} - 2$

$\frac{x^4 - 1}{x + 1} = x^3 - x^2 + x - 1$

$\frac{x^5 - 1}{x + 1} = x^4 - x^3 + x^2 - x + 1, \text{R} - 2$

Divide to check.

$$\begin{array}{r}
x^2 - x + 1 \\
x + 1 \overline{)\, x^3 + 0x^2 + 0x - 1} \\
\underline{x^3 + x^2} \\
-x^2 + 0x \\
\underline{-x^2 - x} \\
x - 1 \\
\underline{x + 1} \\
-2
\end{array}$$

$$\begin{array}{r}
x^3 - x^2 + x - 1 \\
x + 1 \overline{)\, x^4 + 0x^3 + 0x^2 + 0x - 1} \\
\underline{x^4 + x^3} \\
-x^3 + 0x^2 \\
\underline{-x^3 - x^2} \\
x^2 + 0x \\
\underline{x^2 + x} \\
-x - 1 \\
\underline{-x - 1} \\
0
\end{array}$$

Selected Solutions — Chapter 6

$$
\begin{array}{r}
x^4 - x^3 + x^2 - x + 1 \\
x + 1 \overline{)\, x^5 + 0x^4 + 0x^3 + 0x^2 + 0x - 1} \\
\underline{x^5 + x^4} \qquad\qquad\qquad\qquad\quad \\
-x^4 + 0x^3 \qquad\qquad\qquad \\
\underline{-x^4 - x^3} \qquad\qquad\qquad \\
x^3 + 0x^2 \qquad\quad\;\; \\
\underline{x^3 + x^2} \qquad\quad\;\; \\
-x^2 + 0x \quad\; \\
\underline{-x^2 - x} \quad\; \\
x - 1 \\
\underline{x + 1} \\
-2
\end{array}
$$

20. Divide to find another factor.

$$
\begin{array}{r}
4x^2 + 23x + 15 \\
x - 2 \overline{)\, 4x^3 + 15x^2 - 31x - 30} \\
\underline{4x^3 - 8x^2} \qquad\qquad\quad \\
23x^2 - 31x \qquad \\
\underline{23x^2 - 46x} \qquad \\
15x - 30 \\
\underline{15x - 30} \\
0
\end{array}
$$

$4x^3 + 15x^2 - 31x - 30 = (x - 2)(4x^2 + 23x + 15)$

Factor further. Find two integers that add to make 23 and multiply to make 60. The integers are 20 and 3.

$$
\begin{aligned}
4x^3 + 15x^2 - 31x - 30 &= (x - 2)(4x^2 + 20x + 3x + 15) \\
&= (x - 2)(4x(x + 5) + 3(x + 5)) \\
&= (x - 2)(x + 5)(4x + 3)
\end{aligned}
$$

The other factors are $4x + 3$ and $x + 5$.

21. Divide to find another factor.

$$
\begin{array}{r}
2x^2 - x + (k + 5) \qquad\; \\
x + 5 \overline{)\, 2x^3 + 9x^2 + kx - 15} \qquad\; \\
\underline{2x^3 + 10x^2} \qquad\qquad\qquad\quad \\
-x^2 + kx \qquad\qquad\quad \\
\underline{-x^2 - 5x} \qquad\qquad\quad \\
(k + 5)x - 15 \qquad\qquad \\
\underline{(k + 5)x + 5(k + 5)} \\
-15 - 5(k + 5)
\end{array}
$$

For the remainder to be 0,

$$
\begin{aligned}
-15 - 5(k + 5) &= 0 \\
-15 - 5k - 25 &= 0 \\
5k &= -40 \\
k &= -8
\end{aligned}
$$

Selected Solutions — Chapter 6

6 Review, page 393

3. Answers may vary.

For exercise 1: I used the formula $A = 4\pi r^2$, and substituted $r = \frac{3.5}{2}$ m, or 1.75 m.

Then $A = 4\pi(1.75)^2$
$\doteq 38.48$

The surface area is approximately $38.5\ m^2$.

For exercise 2: I used the formula $V = \frac{4}{3}\pi r^3$, and substituted $V = 3.5$ to calculate r.

Then $3.5 = \frac{4}{3}\pi r^3$

$$r^3 = \frac{3(3.5)}{4\pi}$$
$$\doteq 0.8356$$
$$r = \sqrt[3]{0.8356}$$
$$\doteq 0.9419$$

Then I used the formula $A = 4\pi r^2$, and substituted for r from above.

$A \doteq 4\pi(0.9419)^2$
$\doteq 11.148$

The surface area is approximately $11.1\ m^2$.

10. Answers may vary. For part a:

I found the area of the large rectangle, then subtracted the area of the small unshaded rectangle.

$(3x + 2y + x + y)(x + 2y + x + y) - (x + 2y)(x + y)$
$= (4x + 3y)(2x + 3y) - (x + 2y)(x + y)$
$= 8x^2 + 18xy + 9y^2 - (x^2 + 3xy + 2y^2)$
$= 7x^2 + 15xy + 7y^2$

The perimeter is the sum of the lengths of the sides of the shaded region.

Perimeter $= (3x + 2y) + (x + 2y) + (x + y) + (x + y)$
$+ (3x + 2y + x + y) + (x + 2y + x + y)$
$= 3x + 2y + x + 2y + x + y + x + y + 4x + 3y + 2x + 3y$
$= 12x + 12y$

15. Answers may vary. For part a: $4x^2 - 7x + 3$; I found two integers that add to make -7 and multiply to make 12. The integers are -4 and -3.

$4x^2 - 7x + 3 = 4x^2 - 4x - 3x + 3$

I factored pairs of terms.

$4x(x - 1) - 3(x - 1)$

I removed $(x - 1)$ as a common factor.

$(x - 1)(4x - 3)$

Selected Solutions — Chapter 6

6 Cumulative Review, page 395

3. Answers may vary. For part a: The second term is –6 and the third term is 12. I divided the third term by the second term to find the common ratio: $r = \frac{12}{-6}$, or –2.
To find the first term, I divided the second term by the common ratio to get $\frac{-6}{-2}$, or 3. To get the fourth term, I multiplied the third term by the common ratio: (12)(–2) = –24. Similarly, the fifth term is (–24)(–2) = 48. The terms are 3, –6, 12, –24, 48.

9. By definition, a triangle with two equal sides is an isosceles triangle. This triangle has two equal sides, AB = BC. It is an isosceles triangle.

16. b) Answers may vary. For part i: I know two points on the line, P(4, 8) and Q(–3, 6). I found the slope of the line passing through these points.

$$\text{Slope of PQ} = \frac{6-8}{-3-4} = \frac{-2}{-7} = \frac{2}{7}$$

I let T(x, y) be any point on the line.

$$\text{Slope of PT} = \frac{y-8}{x-4}$$

Since the slope of PT = the slope of PQ

$$\frac{y-8}{x-4} = \frac{2}{7}$$

I simplified the equation using the shortcut method.

$$7(y-8) = 2(x-4)$$
$$7y-56 = 2x-8$$

I collected all the terms on the left side.

The equation of the line is $2x - 7y + 48 = 0$.

18. b) Answers may vary. For part iii: Subtract 3 from the input number. The input number is x; so when I subtract 3, I get $x - 3$. This is equal to the output number y. So, the equation is $y = x - 3$.